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A New Triangle Generation of Some Generalized Genocchi Numbers

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Abstract

The two-dimensional rook theory can be generalized to three and higher dimensions by assuming that rooks attack along hyperplanes. Using this generalization, Alayont and Krzywonos defined two separate families of boards in three and higher dimensions generalizing the two-dimensional triangular boards whose rook numbers correspond to generalizations of Stirling numbers of the second kind and Genocchi numbers. This combinatorial interpretation of the Genoc-

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chi numbers was shown to provide a new triangle generation of the Genocchi numbers. In this paper, we prove similar triangle generations for the third and fourth generalized Genocchi numbers using rook numbers of boards in four and five dimensions.

1 Introduction

The theory of rook polynomials (in two dimensions) was developed to provide a way of counting permutations with restricted positions. Consider a non-attacking rook placement on a square board with restricted cells where each row and column has one rook. This placement corresponds to a permutation σ with restrictions by letting a rook placed in row i column j to mean $\sigma(i) = j$. In other words, the permutation is obtained by reading the column numbers of the rooks from top row to bottom. For example, the configuration of non-attacking rooks shown in Fig. 1 corresponds to the permutation (132) in cycle notation, or 312 in word notation. On this board, the restrictions for the permutations, represented with the shaded cells, are $\sigma(1) \neq 1,2$ and $\sigma(2) \neq 2$.

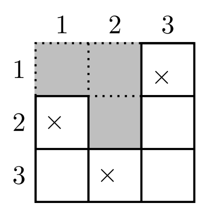

Figure 1: The rook placement corresponding to the permutation (132).

It is known that the numbers of ways to place non-attacking rooks on certain families of boards correspond to well-known number sequences. One such example is the family of triangular boards. The two-dimensional size 5 triangular board is shown in Fig. 2. It is well known that the number of ways to place k non-attacking rooks on a triangular board of size n corresponds to the Stirling numbers of the second kind, $S(n+1, n+1-k)$. Recall that the Stirling number $S(n,k)$ counts the number of ways to partition a set of n elements into k non-empty subsets. We review the proof of this result in Section 2.

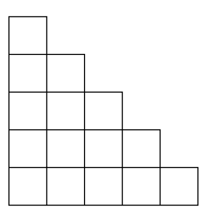

Figure 2: The size 5 triangular board in 2D.

The two-dimensional rook theory can be generalized to higher dimensions by letting rooks attack along hyperplanes [9]. Using this generalization, the two-dimensional triangular boards were generalized to three and higher dimensions in two different ways, yielding families of triangular boards and Genocchi boards [1]. The numbers of ways to place k non-attacking rooks on the three-dimensional triangular boards result in the central factorial numbers [1]. In four and higher dimensions, k non-attacking rooks on the triangular boards generate the generalized central factorial numbers [3]. In this paper, we focus on the rook numbers of the Genocchi boards. Placing the maximum number of rooks on the three-dimensional Genocchi boards corresponds to the $(n+1)$st unsigned Genocchi numbers (of even index) $1,1,3,17,155,2073,38227,929569,\ldots$, and hence the name Genocchi boards [1]. In higher dimensions, the resulting sequence is the generalized Genocchi numbers [3].

Using the representation of the Genocchi numbers as the rook numbers of the three-dimensional Genocchi boards and the recursive nature of these boards, one obtains a combinatorial proof of a new recurrence relation for Genocchi numbers [2]. This recurrence relation is different than the Seidel generation or other triangles mentioned in [4, 6, 7]. This paper expands this recurrence relation of rook numbers of Genocchi boards in three dimensions to four and five dimensions, thus providing triangle generations of the third and fourth generalized Genocchi numbers. Along with these recurrence relations, we will describe how to visualize Genocchi boards in four and five dimensions.

In Section 2, we review the background material including the proof of the triangle generation of the Genocchi numbers in [2] obtained by using rook placements on Genocchi boards in three dimensions. We also include the combinatorial interpretation of Genocchi numbers and the recurrence relation using the rook representation. Then in Sections 3 and 5, we progress to the proof of the recurrence relations of generalized Genocchi numbers using Genocchi boards in four and five dimensions to obtain a triangle generation of the generalization of Genocchi numbers.

2 Two and Three Dimensions

We follow the notation and terminology of rook theory as described in [1]. In two dimensions, a *board* can be visualized as a collection of unit square cells chosen from a large size chess board. Each cell can be represented as (i,j) where i represents the row number (from top to bottom) and j represents the column number (from left to right). Rooks placed on these boards can attack along rows and columns. Therefore, a *non-attacking placement of rooks* on a board means that no two rooks lie in the same row or column.

The *two-dimensional triangular board of size n* consists of ordered pairs (i,j) with $j \le i$ and $1 \le i \le n$, as illustrated in Fig. 2. It is known that the number of placements of k non-attacking rooks on the size n triangular board is equal to $S(n+1,n+1-k)$ where $S(n,k)$ are the Stirling numbers of the second kind. The proof of the relationship uses induction and the fact that the Stirling numbers satisfy the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

with initial values $S(n,1) = 1$ and $S(n,n) = 1$. There are two cases of rook

placements on the triangular board: one where there is no rook on the bottom row and the other where there is a rook on the bottom row. If there is no rook on the bottom row, the k rooks are placed on the top $n-1$ rows and, by induction, there are $S(n,n-k)$ such placements. If there is a rook on the bottom row, then $k-1$ rooks must be placed on the top $n-1$ rows. Thus, there are $S(n,n-k+1)$ ways to place them. Once these rooks have been placed, the last rook will have $n-k+1$ available cells. Therefore, there are $(n-k+1)S(n,n-k+1)$ total choices in this case. Adding the two cases yields $S(n+1,n+1-k)$ using the Stirling number recurrence relation, and, by induction, the claim holds for all n.
Using ideas similar to those in [9], the classical rook theory was generalized to three and higher dimensions in [1] by letting rooks attack along hyperplanes consisting of cells with one fixed coordinate. In three dimensions, the cells are represented as triples (i,j,k) and a rook on this cell will attack cells of the form $(i,*,*),(*,j,*)$, and $(*,*,k)$. We use *wall, slab* and *layer* to refer to the plane of cells with the first, second and third coordinate fixed, respectively. Hence, in three dimensions, rooks attack along walls, slabs and layers. Layers are numbered from top to bottom, and in each fixed layer, the numbering of rows and columns follows the same convention as in the two-dimensional case.
In [1], the two-dimensional triangular board was also generalized to three dimensions to consist of the cells of the form (i,j,k) with $1 \leq i,j \leq k$ and $1 \leq k \leq n$. The size 4 triangular board in three dimensions is pictured in Fig. 3.

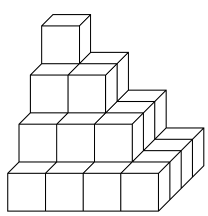

Figure 3: A triangular board in 3D.

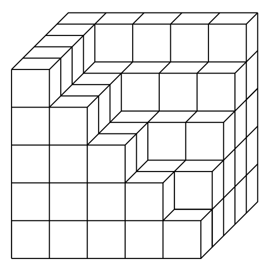

Figure 4: A Genocchi board in 3D.

Notice that Figures 3 and 4 together form a complete size 5 three-dimensional board. Using this relationship, it was shown in [1] that the number of non-attacking placements of n rooks on a size n board as shown in Fig. 4 corresponds to the $(n+1)$st unsigned Genocchi numbers of even index. Consequently, these boards were named the Genocchi boards, denoted $\Gamma_n^{(3)}$ for the dimension being 3. In terms of cell coordinates, $\Gamma_n^{(3)}$ consists of triples of the form (i,j,k), where $1 \leq i,j,k \leq n$ and $\min\{i,j\} \leq k$.
In Fig. 4 showing a size 5 Genocchi board, we can see a smaller size 4 Genocchi board if we remove the back wall and the left slab. Using this inclusion, we can find a recurrence relation between the numbers of rook placements on size n and $n-1$ Genocchi boards as described in the following theorem, proven in [2].

Theorem 1. *Let $r_k(\Gamma_n^{(3)})$ denote the number of ways to place k non-attacking rooks on $\Gamma_n^{(3)}$, the size n Genocchi board in three dimensions. Then $r_k(\Gamma_n^{(3)})$*

satisfies the recurrence relation

$$\begin{aligned} r_k(\Gamma_n^{(3)}) &= r_k(\Gamma_{n-1}^{(3)}) \\ &\quad + r_{k-1}(\Gamma_{n-1}^{(3)})(2(n-k)+1)(n-k+1) \\ &\quad + r_{k-2}(\Gamma_{n-1}^{(3)})(n-k+2)(n-k+1)^3, \end{aligned}$$

where $r_k(\Gamma_n^{(3)}) = 0$ *for* $k < 0$*.*
Below is an outline of the proof given in [2].

Proof. Note that $\Gamma_{n-1}^{(3)}$ can be realized inside $\Gamma_n^{(3)}$ by removing the outside wall (cells with $i = 1$) and slab (cells with $j = 1$). In terms of coordinates, we can represent $\Gamma_{n-1}^{(3)}$ as

$$\{(i,j,k) : 2 \leq i,j,k \leq n \text{ and } \min\{i,j\} \leq k\}.$$

This correspondence works by adding 1 to each coordinate of the triples (i,j,k) in the usual representation of $\Gamma_{n-1}^{(3)}$, which is

$$\Gamma_{n-1}^{(3)} = \{(i,j,k) : 1 \leq i,j,k \leq n-1 \text{ and } \min\{i,j\} \leq k\}.$$

The k rook placements on $\Gamma_n^{(3)}$ can then be split into three possible cases: 1) all k rooks are placed on $\Gamma_{n-1}^{(3)}$ inside $\Gamma_n^{(3)}$, or 2) $k-1$ rooks are placed on $\Gamma_{n-1}^{(3)}$ and one on the outside wall or slab, or 3) $k-2$ rooks are placed on $\Gamma_{n-1}^{(3)}$ and two on the outside wall and slab, except on their intersection. The terms in the recurrence relation correspond to each of these cases in order. The r_i terms, rook numbers, count the number of ways of placing i rooks on $\Gamma_{n-1}^{(3)}$. The factors multiplying the rook numbers are the number of choices for placing the remaining one or two rooks on the outside wall and/or slab. Once the majority of the rooks are placed on the inside $\Gamma_{n-1}^{(3)}$, corresponding rows and columns are deleted from the outside wall and slab, leaving us with a subset of cells that the remaining one or two rooks can be placed. The number of such cells are counted by the factors in front of r_i terms in the recurrence relation. □

From the proof, we notice that the recurrence relation holds due to the nature of the difference $\Gamma_n^{(3)} \setminus \Gamma_{n-1}^{(3)}$. Because this difference is composed of a full wall and a slab, where the rooks are placed on $\Gamma_{n-1}^{(3)}$ does not affect the choices for the remaining rooks. Also note that while this geometric proof works well in the three-dimensional case and provides us the intuition to notice the existence of a recurrence relation for the rook numbers of $\Gamma_n^{(3)}$ and how to prove this relation, it becomes challenging in higher dimensions. For this reason, we provide another proof of Theorem 1 via a symbolic approach.

Proof. (Symbolic approach) Again, we consider three cases: 1) all k rooks are placed on $\Gamma_{n-1}^{(3)}$ inside $\Gamma_n^{(3)}$, or 2) $k-1$ rooks are placed on $\Gamma_{n-1}^{(3)}$ and one on the outside wall or slab, or 3) $k-2$ rooks are placed on $\Gamma_{n-1}^{(3)}$, two on the outside wall and slab.
In the first case, there are $r_k(\Gamma_{n-1}^{(3)})$ ways to place the rooks.
In the second case, we first place the $k-1$ rooks on $\Gamma_{n-1}^{(3)}$ in $r_{k-1}(\Gamma_{n-1}^{(3)})$ ways. Then

the last rook can be placed on the outside wall, cells of the form $(1,*,*)$, or on the outside slab, $(*,1,*)$. However, the previous $k-1$ rooks make $k-1$ coordinates ineligible in each of the free positions, leaving us with $2(n-k+1)^2-(n-k+1)$ cells. We subtract $n-k+1$ cells of the form $(1,1,*)$ since they were double counted. This number is exactly the coefficient of $r_{k-1}(\Gamma_{n-1}^{(3)})$ in the recurrence relation we want to show.
In the third case when $k-2$ rooks are on $\Gamma_{n-1}^{(3)}$, the two remaining rooks are placed on the outside wall and slab. Since we cannot place both on the wall or both on the slab, we must place one on the wall and one on the slab. For the rook on the wall $(1,*,*)$, we have $(n-k+1)(n-k+2)$ choices. This is because each of the previous $k-2$ rooks eliminates one coordinate in each position. We also eliminate 1 from the second position since that would place the rook on the intersection of the wall and the slab. For the rook on the slab $(*,1,*)$, we have $(n-k+1)$ choices for each coordinate due to eliminations from the previous $k-1$ rooks. Therefore, we have a total of $(n-k+2)(n-k+1)^3$ choices for the last two rooks, and this is the coefficient of $r_{k-2}(\Gamma_{n-1}^{(3)})$ in the recurrence relation. The sum of all cases yields the recurrence relation. □

In the special case of $k=n$, this algebraic recurrence relation has a combinatorial interpretation. In [3], $r_n(\Gamma_n^{(3)})$ are interpreted as pairs of permutations (π_1,π_2) of n where $\pi_1(i)\le i$ or $\pi_2(i)\le i$. This correspondence is obtained by generalizing the permutation correspondence described in the Introduction. In three dimensions, rooks are placed in cells (i,j,k). Suppose there are n rooks on $\Gamma_n^{(3)}$. Because of the way rooks attack, this means there is exactly one rook per layer. If we order the rook positions so that their last coordinate is increasing, then we obtain two permutations by reading the first coordinates and the second coordinates. In other words, $\pi_1(k)=i,\pi_2(k)=j$ if the rook is in cell (i,j,k).
Since the permutation interpretation requires a rook in each layer, we will use a modification of permutations to interpret $r_{n-2}(\Gamma_{n-1}^{(3)})$ combinatorially. We use partial permutations, specifically those with one hole, to account for the fact that one layer will not have a rook [5]. A *partial permutation of r with one hole* is a bijection between two size $r-1$ subsets of $\{1,2,\dots,r\}$. It can be represented as a string of $r-1$ distinct numbers chosen from 1 to r and $\lozenge$, representing the hole. Using the same idea as when we placed n rooks on $\Gamma_n^{(3)}$, we can interpret $r_{n-2}(\Gamma_{n-1}^{(3)})$ as pairs (π_1,π_2) of partial permutations of $n-1$ with one hole in the same place so that $\pi_1(i)\le i$ or $\pi_2(i)\le i$ when i is not the hole. Then the algebraic recurrence relation translates into a recurrence relation for the number of pairs of permutations of n expressed in terms of the numbers of pairs of permutations of $n-1$ and pairs of partial permutations of $n-1$ with one hole in the same place, all pairs with the condition that $\pi_1(i)\le i$ or $\pi_2(i)\le i$.
Using similar ideas as in the symbolic proof of Theorem 1, we will obtain recurrence relations and their combinatorial interpretations for four and five dimensions in the next two sections.

3 Four Dimensions

The three-dimensional Genocchi boards were generalized to four and higher dimensions in [3]. In four and higher dimensions, rooks attack along hyperplanes, which consist of cells with one fixed coordinate. In four dimensions, we use *walls, slabs, layers* and the *time* to indicate the hyperplanes of cells with fixed 1^{st}, 2^{nd}, 3^{rd} and 4^{th} coordinates, respectively. The cells of $\Gamma_n^{(4)}$, the size n four-dimensional Genocchi board, are tuples of the form (i,j,k,ℓ), where $1 \le i,j,k,\ell \le n$ and $\min\{i,j,k\} \le \ell$. One way to visualize four dimensional Genocchi boards is to look at the board at a fixed time, i.e. with fixed last coordinate values. The size 3 Genocchi board in four dimensions is shown below, with each piece representing a different time value.

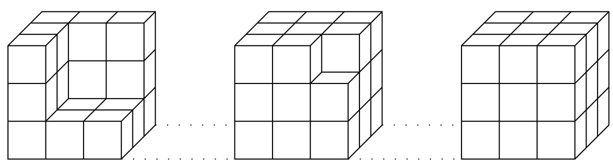

Figure 5: Size 3 Genocchi board in four dimensions.

Note that for ease of visualization, in four dimensions, we number layers (k coordinate) from bottom to top, and time (ℓ) from left to right. Thus the leftmost piece of the board above corresponds to cells (i,j,k,ℓ) with $\ell = 1$ and i or j or $k = 1$.

Similar to the recursive nature of the boards in two and three dimensions, $\Gamma_n^{(4)}$ includes the board $\Gamma_{n-1}^{(4)}$, identified as

$$\Gamma_{n-1}^{(4)} = \{(i,j,k,\ell) : 2 \le i,j,k,\ell \le n \text{ and } \min\{i,j,k\} \le \ell\}.$$

We remove the cells with coordinates $i = 1$, $j = 1$, or $k = 1$ from the larger Genocchi board to obtain the smaller board. In Fig. 5 showing $\Gamma_3^{(4)}$, this corresponds to removing all the outside walls in the back (corresponding to $i = 1$), outside slabs on the left (corresponding to $j = 1$) and outside layers at the bottom (corresponding to $k = 1$). Thus, we can visualize $\Gamma_2^{(4)}$ as the size 2 cube at the front top right corner of the rightmost size 3 cube and the size 2 cube with one cell missing at the same place of the middle size 3 cube.

More generally, the difference $\Gamma_n^{(4)} \backslash \Gamma_{n-1}^{(4)}$ consists of one outside wall corresponding to cells with $i = 1$, one outside slab for cells with $j = 1$ and one outside layer corresponding to cells with $k = 1$. The cells with coordinates $(1,1,1,*)$ belong to the intersection of the outside wall, slab and layer, which we call the ***triple intersection***. Any pair of wall, slab, or layer share n cells of the form $(1,1,*,*)$, or $(1,*,1,*)$, or $(*,1,1,*)$, which we call ***pair intersections***. When placing more than one rook on the difference $\Gamma_n^{(4)} \backslash \Gamma_{n-1}^{(4)}$, we must be careful about the intersections.

The relationship between $\Gamma_n^{(4)}$ and $\Gamma_{n-1}^{(4)}$ allows us to derive a recurrence relation for the rook numbers, as demonstrated by the following theorem.

Theorem 2. *Let $r_k(\Gamma_n^{(4)})$ denote the number of ways to place k non-attacking rooks on the size n four dimensional Genocchi board. Then $r_k(\Gamma_n^{(4)})$ satisfies the* *recurrence relation*

$$\begin{aligned} r_k(\Gamma_n^{(4)}) &= r_k(\Gamma_{n-1}^{(4)}) \\ &+ r_{k-1}(\Gamma_{n-1}^{(4)})(n-k+1)(3(n-k+1)^2 - 3(n-k+1)+1) \\ &+ r_{k-2}(\Gamma_{n-1}^{(4)})(n-k+2)3(n-k+1)^5 \\ &+ r_{k-3}(\Gamma_{n-1}^{(4)})(n-k+3)(n-k+2)^4(n-k+1)^4, \end{aligned}$$

where $r_k(\Gamma_n^{(4)}) = 0$ *for* $k < 0$ *and* $r_0(\Gamma_n^{(4)}) = 1$.

Proof. The number of ways to place k non-attacking rooks on $\Gamma_n^{(4)}$ can be partitioned into 4 cases: 1) all k rooks are placed on $\Gamma_{n-1}^{(4)}$ realized inside $\Gamma_n^{(4)}$, 2) $k-1$ rooks are placed on $\Gamma_{n-1}^{(4)}$, 3) $k-2$ rooks are placed on $\Gamma_{n-1}^{(4)}$, and 4) $k-3$ rooks are placed on $\Gamma_{n-1}^{(4)}$.

In the first case, there are $r_k(\Gamma_{n-1}^{(4)})$ ways to place the rooks.

In the second case, there are $r_{k-1}(\Gamma_{n-1}^{(4)})$ possible placements for $k-1$ rooks inside $\Gamma_{n-1}^{(4)}$. Once $k-1$ rooks are placed on $\Gamma_{n-1}^{(4)}$, there is one rook to place on the outside wall, slab, or layer. Each has $(n-k+1)^3$ cells eligible due to the previous placement of $k-1$ rooks on $\Gamma_{n-1}^{(4)}$ disqualifying $k-1$ of the n available coordinates in two free positions, leading to a total of $3(n-k+1)^3$ options. However, this approach does not count the intersections properly. Therefore, we use the *Inclusion-Exclusion Principle* to obtain the result. Altogether, we have $r_{k-1}(\Gamma_{n-1}^{(4)})(n-k+1)(3(n-k+1)^2 - 3(n-k+1)+1)$ ways to place k rooks on $\Gamma_n^{(4)}$ where $k-1$ rooks lie on $\Gamma_{n-1}^{(4)}$.

In the third case, there are $r_{k-2}(\Gamma_{n-1}^{(4)})$ ways to place $k-2$ rooks on $\Gamma_{n-1}^{(4)}$. Note that in this case we cannot place either of the final two rooks on the triple intersection since a rook on the triple intersection disqualifies all the remaining cells on the outside wall, slab, and layer for the second rook. However, the pair intersections can be used and we consider this case separately. We have two subcases: 3.1) both rooks on outside wall, slab, or layer excluding the pair intersections, and 3.2) one on a pair intersection and the other on the non-attacked wall, slab, or layer.

For 3.1), we first choose which two of the wall, slab, and layer to use for the final two rooks, which can be done in 3 ways. Let us suppose we chose wall and slab to use. On the wall, $(1,*,*,*)$, we have $(n-k+2)$ choices for the last coordinate due to the previous $k-2$ rook. We then have $(n-k+1)$ choices for each of the middle two coordinates since we cannot use 1 and the options eliminated by the previous $k-2$ rooks. After placing this rook, for the last rook, we can use the slab cells $(*,1,*,*)$. For these, we have $(n-k+1)^2(n-k)$ choices. For this subcase, we obtain a total of $3(n-k+2)(n-k+1)^4(n-k)$ choices.

For 3.2), we first choose which pair intersection to use for one of the final two rooks, and there are 3 options. Once we decide on the pair intersection, only one of the wall, slab, or layer will be available to use for the second rook. Say we chose the pair intersection of the wall and slab, $(1,1,*,*)$. We have $(n-k+2)(n-k+1)$ cells available due to the previous $k-2$ rooks places and 1 not

being available in the third position. Then on the remaining layer $(*,*,1,*)$, we have $(n-k+1)^3$ options. Putting these together gives us $3(n-k+2)(n-k+1)^4$ options.
Since 3.1) and 3.2) are mutually exclusive cases we add them to obtain $r_{k-2}(\Gamma^{(4)}_{n-1})3(n-k+2)(n-k+1)^4\big((n-k)+1\big)$ ways to place k rooks on $\Gamma^{(4)}_n$ where $k-2$ rooks lie on $\Gamma^{(4)}_{n-1}$, which simplifies to $r_{k-2}(\Gamma^{(4)}_{n-1})(n-k+2)3(n-k+1)^5$.
In the fourth case, there are $r_{k-3}(\Gamma^{(4)}_{n-1})$ ways to place $k-3$ rooks on $\Gamma^{(4)}_{n-1}$. In this case we cannot place any of our final rooks on the triple intersection or on the pair intersections. Therefore, one rook is on the wall, one on the slab, and one on the layer. For the cells $(1,*,*,*)$, we have $(n-k+3)(n-k+2)^2$ options due to the previous $k-3$ rooks and 1 not being available in the second or third coordinates. For the cells $(*,1,*,*)$, we have $(n-k+2)^2(n-k+1)$ options due to the previous $k-2$ rooks and 1 not being available in the third coordinate. In this case, we do not reduce our options for the first coordinate since the rook on the wall used the 1 option. Finally, on the layer, we have $(n-k+1)^3$ options. Putting all of this together, we obtain $r_{k-3}(\Gamma^{(4)}_{n-1})(n-k+3)(n-k+2)^4(n-k+1)^4$ ways to place k rooks on $\Gamma^{(4)}_n$ where $k-3$ rooks lie on $\Gamma^{(4)}_{n-1}$.
Combining all these cases together, we obtain the recurrence relation in general.
□

Using the above recurrence relation, we obtain Table 1 providing the number of ways to place k non-attacking rooks on $\Gamma^{(4)}_n$.

$n\backslash k$	0	1	2	3	4	5
1	1	**1**				
2	1	15	**7**			
3	1	72	505	**145**		
4	1	220	7525	33135	**6631**	
5	1	525	55445	1207260	3778201	**566641**

Table 1: The coefficients $r_k(\Gamma^{(4)}_n)$.

Notice that as was shown before, the highlighted entries follow the sequence A064624 in [8]: *Generalization of the Genocchi numbers given by the Gandhi polynomials*

$$A(n+1,r) = r^3 \cdot A(n,r+1) - (r-1)^3 \cdot A(n,r); A(1,r) = r^3 - (r-1)^3.$$

Using the combinatorial interpretation of non-attacking rooks on a Genocchi board in the special case of $k = n$, the algebraic recurrence relation translates into a recurrence relation for the number of triples of permutations of n expressed in terms of the numbers of triples of permutations of $n-1$, triples of partial permutations of $n-1$ with one hole in the same place, and triples of partial permutations of $n-1$ with two holes in the same places. For all permutations and partial permutations considered, we require the condition that $\pi_1(i) \leq i$ or $\pi_2(i) \leq i$ or $\pi_3(i) \leq i$.

4 Five Dimensions

The three and four dimensional Genocchi boards can be generalized to five dimensions. In five dimensions, we use *walls, slabs, layers, time* and *hyper-time* to indicate the hyperplanes of cells with fixed 1^{st}, 2^{nd}, 3^{rd}, 4^{th} and 5^{th} coordinates respectively. The cells of $\Gamma_n^{(5)}$, the size n five-dimensional Genocchi board, are tuples of the form (i,j,k,l,h), where $1 \le i,j,k,l,h \le n$ and $\min\{i,j,k,l\} \le h$.
One way to visualize a 5-dimensional Genocchi board is to look at the board with the last two coordinates fixed. The size 3 Genocchi board in five dimensions is shown in Fig. 6.

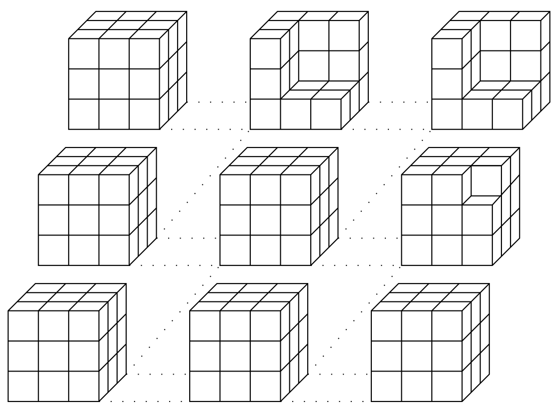

Figure 6: Size 3 Genocchi board in five dimensions.

We follow the same convention as in four dimensions within each row: rows are numbered from back to front, columns left to right, layers bottom to top, and time from left to right. The hyper-time increases from top to bottom. In the bottom row when $h = 3$, due to the definition, the coordinates i,j,k,ℓ can take any of the values 1, 2, 3. In the middle row, when $h = 2$, the coordinates i,j,k are allowed to be 1, 2, 3 when $\ell = 1,2$, but when $\ell = 3$, we need $\min\{i,j,k\} = 2$. Therefore, when $h = 2, \ell = 3$, the cell $(3,3,3,3,2)$ is missing. When $h = 1$, the coordinates i,j,k are allowed to be 1, 2, 3 when $\ell = 1$, and otherwise, at least one of i,j,k is 1.
Similar to the four-dimensional case, $\Gamma_n^{(5)}$ includes the smaller board $\Gamma_{n-1}^{(5)}$, which can be formally defined as

$$\Gamma_{n-1}^{(5)} = \{(i,j,k,l,h) : 2 \le i,j,k,l,h \le n \text{ and } \min\{i,j,k,l\} \le h\}.$$

The difference $\Gamma_n^{(5)} \backslash \Gamma_{n-1}^{(5)}$ consists of one outside slab with cells $(1,*,*,*,*)$ on $\Gamma_n^{(5)}$, one outside wall with cells $(*,1,*,*,*)$, one outside layer with cells $(*,*,1,*,*)$, and one outside time with cells $(*,*,*,1,*)$. The outside wall, slab, layer and time share the cells $(1,1,1,1,*)$, the ***quadruple intersection***. Intersections of three are called the ***triple intersections*** and the intersections of any pair of wall, slab, or layer are the ***pair intersections***.

The relationship between $\Gamma_n^{(5)}$ and $\Gamma_{n-1}^{(5)}$ allows us to derive a recurrence relation for the rook numbers, as demonstrated by the following theorem.

Theorem 3. *Let $r_k(\Gamma_n^{(5)})$ denote the number of ways to place k non-attacking rooks on the size n five dimensional Genocchi board. Then $r_k(\Gamma_n^{(5)})$ satisfies the recurrence relation*

$$\begin{aligned} r_k(\Gamma_n^{(5)}) &= r_k(\Gamma_{n-1}^{(5)}) \\ &+ r_{k-1}(\Gamma_{n-1}^{(5)})(n-k+1)\sum_{i=1}^{4}(-1)^{i+1}\binom{4}{i}(n-k+1)^{4-i} \\ &+ r_{k-2}(\Gamma_{n-1}^{(5)})(n-k+2)(n-k+1)^5(6(n-k+1)^2+1) \\ &+ r_{k-3}(\Gamma_{n-1}^{(5)})(n-k+3)(n-k+2)^5(n-k+1)^5(4(n-k+1)+2) \\ &+ r_{k-4}(\Gamma_{n-1}^{(5)})(n-k+4)\prod_{i=1}^{3}(n-k+i)^5, \end{aligned}$$

where $r_k(\Gamma_n^{(5)}) = 0$ for $k < 0$ and $r_0(\Gamma_n^{(5)}) = 1$.

Proof. The number of ways to place k non-attacking rooks on the five dimensional Genocchi board size n can be separated into 5 possible cases: 1) all k rooks are placed on $\Gamma_{n-1}^{(5)}$ realized inside $\Gamma_n^{(5)}$, 2) $k-1$ rooks are placed on $\Gamma_{n-1}^{(5)}$, 3) $k-2$ rooks are placed on $\Gamma_{n-1}^{(5)}$, 4) $k-3$ rooks are placed on $\Gamma_{n-1}^{(5)}$, or 5) $k-4$ rooks are placed on $\Gamma_{n-1}^{(5)}$.

In the first case, there are $r_k(\Gamma_{n-1}^{(5)})$ ways to place the rooks.

In the second case, there are $r_{k-1}(\Gamma_{n-1}^{(5)})$ ways to place $k-1$ rooks on $\Gamma_{n-1}^{(5)}$. Once $k-1$ rooks are placed on $\Gamma_{n-1}^{(5)}$, there is one rook to place on the outside wall, slab, layer, or time. Similar to the 4D proof, we use the *Inclusion-Exclusion Principle* to count available cells to be $4(n-k+1)^4-6(n-k+1)^3+4(n-k+1)^2-(n-k+1)$. Thus, there are $\Gamma_n^{(5)}(n-k+1)(4(n-k+1)^3-6(n-k+1)^2+4(n-k+1)-1)$ ways to place k rooks on $\Gamma_n^{(5)}$ where $k-1$ rooks lie on $\Gamma_{n-1}^{(5)}$. We can also count this case by noting that the number of cells is found by excluding the cells where none of the first four coordinates are 1. There are $(n-k+1)^4-(n-k)^4$ options for the first four coordinates, and $(n-k+1)$ options for the last. This gives us a total of $r_{k-1}(\Gamma_{n-1}^{(5)}(n-k+1)((n-k+1)^4-(n-k)^4))$, which equals the expression given in the recurrence relation above.

In the third case, we place $k-2$ rooks on $\Gamma_{n-1}^{(5)}$ and two on the outside. We split how to place the last two rooks into four subcases: 3a) neither rook on an intersection, with only allowed cells being $(1,*,*,*,*),(*,1,*,*,*),(*,*,1,*,*)$, (*, *, *, 1, *) where no $*$ is 1; 3b) one rook on a pair intersection, one on no intersection; 3c) both rooks on pair intersections; and 3d) one rook on a triple intersection and one not. There are 6 possibilities for the subcase 3a) based on which two coordinates have 1's. After choosing which possibility we have, we have $n-k+2$ choices for the hyper-time coordinate of the second to last rook and every other coordinate besides the coordinate with 1 has $n-k+1$ choices. For the final rook, we have $n-k+1$ hyper-time choices. For the coordinate that corresponds to where 1 is in the second to last rook, we again have $n-k+1$ choices. For the two other remaining free coordinates, we have $n-k$ choices

each. Therefore, each possibility has $(n-k+2)(n-k+1)^5(n-k)^2$ options. There are 12 possibilities for the subcase 3b), each with $(n-k+2)(n-k+1)^5(n-k)$ options using a similar coordinate option counting. There are 3 possibilities for the subcase 3c), each with $(n-k+2)(n-k+1)^5$ options. Finally, 3d) has 4 possibilities and each has $(n-k+2)(n-k+1)^5$. Adding all the options gives $(n-k+2)(n-k+1)^5(6(n-k)^2+12(n-k)+7)$, which is the coefficient of $r_{k-2}(\Gamma_{n-1}^{(5)})$ given in the recurrence relation above.
In the fourth case, we split how to place the last three rooks into two subcases: 4a) none of the rooks is on an intersection; and 4b) one rook on a pair intersection including time intersections, and the other two on no intersections. There are 4 possibilities for the subcase 4a based on which coordinate is never 1. After choosing which possibility we have and ordering the rooks arbitrarily based on where the 1's are, we have $n-k+3$ choices for the hyper-time coordinate of the third to last rook and every other coordinate besides the coordinate with 1 has $n-k+2$ choices. For the middle rook, we have $n-k+2$ choices for hyper-time, $n-k+2$ choices for the coordinate that overlaps with the coordinate 1 of the third to last rook, and $n-k+1$ choices for each of the other two coordinates. For the final rook, we have $n-k+1$ hyper-time choices. For the two coordinates that correspond to where 1 was in the previous two rooks, we again have $n-k+1$ choices. For the last coordinate, we have $n-k$ choices. Therefore, each possibility has $(n-k+3)(n-k+2)^5(n-k+1)^5(n-k)$ options. There are 6 possibilities for the subcase 4b each with $(n-k+3)(n-k+2)^5(n-k+1)^5$ options using a similar coordinate option counting. Adding all the options gives $(n-k+3)(n-k+2)^5(n-k+1)^5(4(n-k)+6)$.
In the fifth case, there are $r_{k-4}(\Gamma_{n-1}^{(5)})$ ways to place $k-4$ rooks on $\Gamma_{n-1}^{(5)}$. Therefore, there are four rooks left that to place on the outside wall, slab, layer or time. Note in this case we cannot place any of these final rooks on an intersection. Therefore, all must be placed on cells of the form $(1,*,*,*,*),(*,1,*,*,*)$, (*, *, 1 , *, *), (*, *, *, 1, *) where no $*$ is 1 and there must be one rook on each of the outside wall, slab, layer, and the time. Counting options for each coordinate as in the previous cases, we find that there are $r_{k-4}(\Gamma_{n-1}^{(5)})(n-k+4)(n-k+3)^3(n-k+3)^2(n-k+2)^2(n-k+2)^2(n-k+1)(n-k+2)(n-k+1)^2(n-k+1)^2$ ways to place k rooks on $\Gamma_n^{(5)}$ where $k-4$ rooks lie on $\Gamma_{n-1}^{(5)}$. This can be simplified to $r_{k-4}(\Gamma_{n-1}^{(5)})(n-k+4)(n-k+3)^5(n-k+2)^5(n-k+1)^5$.
Putting all these cases together, we obtain the recurrence relation in general.
□

Using the above recurrence relation, we obtain Table 2 that gives the number of ways to place k non-attacking rooks on the size n five dimension Genocchi board.
As expected, the highlighted entries follow the sequence A064625 in [8]: *Generalization of the Genocchi numbers given by the Gandhi polynomials*

$$A(n+1,r)=r^4\cdot A(n,r+1)-(r-1)^4\cdot A(n,r);A(1,r)=r^4-(r-1)^4.$$

Similar to the four-dimensional case, when $k=n$, the algebraic recurrence relation above translates into a recurrence relation for the number of quadruples of permutations of n expressed in terms of the numbers of the quadruples of

$n\backslash k$	0	1	2	3	4	5
1	1	**1**				
2	1	31	**15**			
3	1	226	3345	**1025**		
4	1	926	100875	954815	**209135**	
5	1	2771	1245715	87547640	598789745	**100482849**

Table 2: The coefficients $r_k(\Gamma_n^{(5)})$.

permutations of $n-1$, quadruples of partial permutations of $n-1$ with one hole in the same place, quadruples of partial permutations of $n-1$ with two holes in the same place and quadruples of partial permutations of $n-1$ with three holes. In all cases, we require that $\pi_1(i) \leq i$ or $\pi_2(i) \leq i$ or $\pi_3(i) \leq i$ or $\pi_4(i) \leq i$.

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