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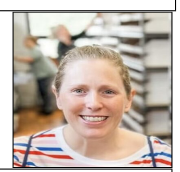
Triangles and variance of the distance to the boundary

Alastair Fletcher , Katherine Fletcher, Joseph Wasiqi*

Alastair Fletcher is a Professor at Northern Illinois University and currently Director of Undergraduate Studies. He finds assisting students at all levels and of all backgrounds fulfilling, and puts up with the department making him wear Halloween costumes.

Katherine Fletcher holds a Master's in Mathematics, teaches at Elgin Community College and is Director of the MathLab. She enjoys helping students reach their potential.



Joseph Wasiqi is currently a Master's degree student at Northern Illinois University. He hopes to carry his mathematical knowledge into medicine and/or bioinformatic.

Abstract

We consider the variance of the distance to the boundary for planar triangles. Our main result is that if γ is a line segment joining a vertex of a triangle to a point on the opposite side, then the variance restricted to γ is a convex function.

1 Introduction

Let U be a compact convex set in $\mathbb{R}^2$. For $z = (x,y)$ contained in U and $\theta \in [0,2\pi)$, we define $d_z^U(\theta)$ to be the distance from z to ∂U in the direction θ. Then for $k \in \mathbb{N}$ we define

$$I_k^U(z) = \frac{1}{2\pi}\int_0^{2\pi} [d_z^U(\theta)]^k \, d\theta.$$

***Corresponding author**: afletcher@niu.edu

Finally, the variance from z to ∂U is defined by

$$\mathrm{var}_U(z) = I_2^U(z) - [I_1^U(z)]^2. \tag{1. A}$$

If the context is clear, we may drop the U from the above notation and just write $d_z(\theta), I_k(z)$ and so on.

The notion of variance of the distance to the boundary was used in Strawbridge et al [4] as the basis of a robust algorithm to determine whether individual cells in an embryo are interior or exterior cells based purely on knowing the location of the nuclei of the cells. In particular, it was shown in [4] that in the special case of the unit ball in $\mathbb{R}^3$, the variance of the distance to the boundary has a unique minimum at the center of the ball. In this paper, we initiate a mathematical study of the variance by focusing on the case of planar triangles. In Figures 1 and 2, we see a plot of the variance function inside a certain isosceles triangle.

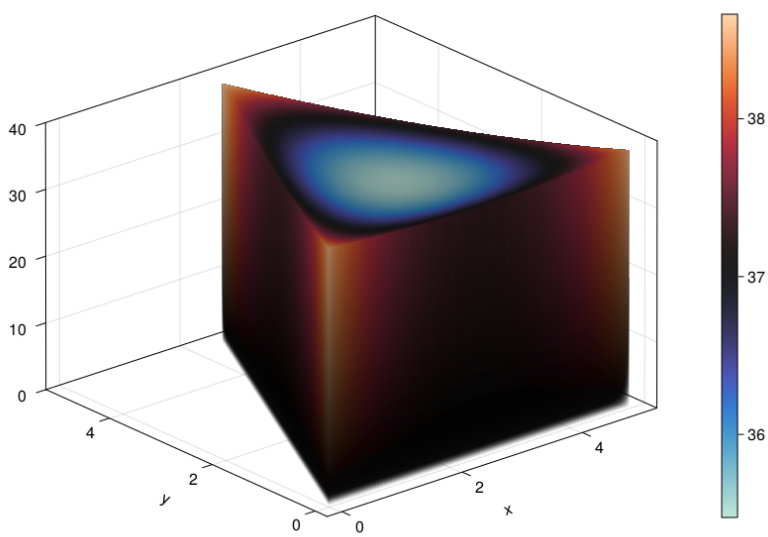


Figure 1: Surface plot generated with 80199 variance values with step size for x and y coordinates equal to 0.0125.

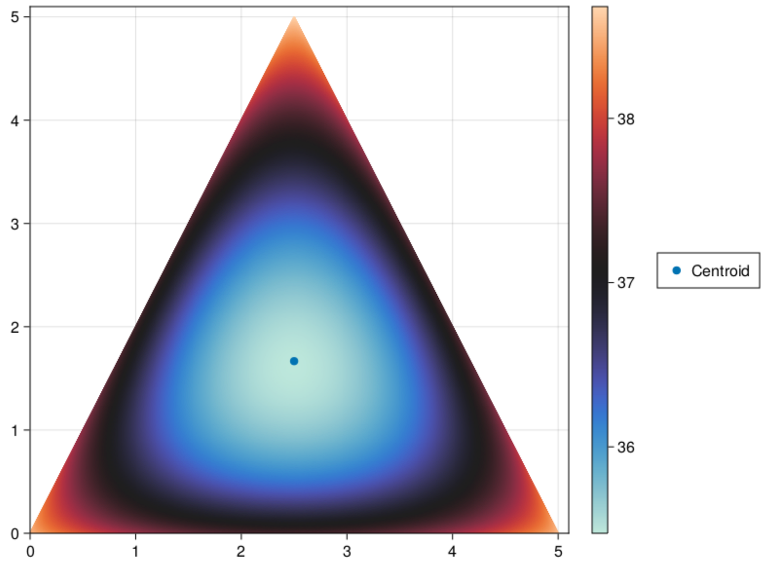


Figure 2: Flattened version of Figure 1 with the centroid of the triangle marked.

We start by giving some general properties of variance.

Theorem 1. *Let U be a compact convex set in $\mathbb{R}^2$.*

(i) If $f:\mathbb{R}^2\to\mathbb{R}^2$ is an isometry, then

$$var_{f(U)}(f(z)) =_U (z).$$

(ii) If $f:\mathbb{R}^2\to\mathbb{R}^2$ is a similarity with scaling factor $C>0$, then

$$var_{f(U)}(f(z)) = C^2_U(z).$$

This result will allow us to pass from general triangles to triangles with vertices at $(0,0),(1,0)$ and (P,Q) with $Q>0$. Additionally, this result also implies that if U has a certain symmetry, then the variance function will also have the analogous symmetry. For example, in an isosceles triangle, the variance is unchanged under a reflection about the line of symmetry. Next, we make the important observation that in two dimensions, $I_2(z)$ is in fact constant in z.

Theorem 2. *For any bounded convex domain $U\subset\mathbb{R}^2$ and any $z\in U$ we have*

$$I_2(z) = \frac{1}{\pi}\,\mathrm{Area}(U).$$

Our main result is as follows.

Theorem 3. *Let T be a triangle and let $\gamma:[0,1]\to T$ parameterize a straight line segment from one vertex of T to a point on the side opposite this vertex. Then $|_\gamma$ is a convex function.*

In particular, it will follow from Theorem 3 that on each line segment emanating from a vertex of T, there is a unique minimum of the variance. Unfortunately, this does not appear to be quite enough to guarantee that the variance is a convex function on the whole of T and thus that the variance has a unique minimum on T. If we could show that the variance is convex on any line segment joining two points on the boundary of T, then this would be enough. However, a typical such line segment will divide T into a triangle and a quadrilateral. Computations analogous to those in this paper for quadrilaterals get messy very quickly. We leave these questions open as a topic for future work, and just note that Figures 2 and 4 below give evidence of a unique minimum.

The paper is organized as follows. In Section 2 we prove the preliminary results Theorem 1 and Theorem 2, while in Section 3 we prove our main result, Theorem 3. We make some concluding remarks and acknowledgements in Section 4.

2 Preliminaries

In this section, we first show that the variance is well behaved under isometries and similarities.

Proof of Theorem 1. Suppose first that f is a translation, rotation or reflection. Then it is clear that I_1 and I_2 are unchanged by applying f and so the same is true of the variance. If f is a scaling by factor $d > 0$, then we have

$$d_{f(z)}^{f(U)}(\theta) = C \cdot d_z^U(\theta),$$

which proves (b) in this case.

For the general case, an isometry is a composition of translations, rotations and reflections, which proves (a). For (b), a similarity is a composition of isometries and scalings as above, which completes the proof. □

Next we prove that I_2 is constant for bounded convex planar domains.

Proof of Theorem 2. For this proof, we will work in complex coordinates. Suppose first that ∂U is smooth and that we parameterize ∂U by fixing $z \in \overline{U}$ and setting $w = \gamma(\theta) = d_z(\theta)e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Then γ is a smooth function and

$$\gamma'(\theta) = (d_z'(\theta) + i d_z(\theta))e^{i\theta}.$$

As $d_z(\theta)$ is real-valued, we have

$$\begin{aligned}\int_\gamma \overline{w}\, dw &= \int_0^{2\pi} \overline{(d_z(\theta)e^{i\theta})}(d_z'(\theta) + i d_z(\theta))e^{i\theta})\, d\theta \\ &= \int_0^{2\pi} (d_z(\theta)d_z'(\theta) + i(d_z(\theta))^2)\, d\theta.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int_0^{2\pi} d_z(\theta)d_z'(\theta)\, d\theta &= \left[d_z(\theta)^2\right]_{\theta=0}^{2\pi} - \int_0^{2\pi} d_z(\theta)d_z'(\theta)\, d\theta \\ &= 0 - \int_0^{2\pi} d_z(\theta)d_z'(\theta)\, d\theta.\end{aligned}$$

Thus

$$\int_0^{2\pi} d_z(\theta)d_z'(\theta)\, d\theta = 0$$

and we have

$$\int_\gamma \overline{w}\, dw = i\int_0^{2\pi} d_z(\theta)^2\, d\theta = 2\pi i I_2(z).$$

By Green's Theorem, $\int_\gamma \overline{w}\, dw$ is nothing other than $2i$ multiplied by the area enclosed by γ, from which Theorem 2 follows in the smooth case.

For the general case, fix $z \in U$ and suppose that $\varphi : \mathbb{D} \to U$ is a conformal bijection from the unit disk onto U given by the Riemann Mapping Theorem. For $0 < r < 1$, set S_r to be the circle centred at the origin of radius r, D_r to be the open disk centred at the origin of radius r, and set $U_r = \varphi(D_r)$. We may assume that r_0 is chosen close enough to 1 that $z \in U_r$ for $r > r_0$. By a theorem of Study [5], see also for example [3], it follows that U_r is a convex domain for all $0 < r < 1$. Moreover, as $\varphi(S_r)$ is a smooth Jordan curve it follows by the argument above that for $r_0 < r < 1$,

$$I_2^{U_r}(z) = \frac{1}{\pi}\,\mathrm{Area}(U_r).$$

As ∂U is a Jordan curve, by Carathéodory's Theorem it follows that φ extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{U}$. Thus the extended function is uniformly continuous, as it is continuous on a compact subset of $\mathbb{C}$. It follows that $d_z^{U_r}(\theta) \to d_z^U(\theta)$ as $r \to 1$ uniformly in θ. A standard integral estimate then shows that $I_2^{U_r}(z) \to I_2^U(z)$ as $r \to 1$. Putting this all together, we have

$$I_2^U(z) = \lim_{r\to 1} I_2^{U_r}(z) = \lim_{r\to 1} \frac{1}{\pi}\,\mathrm{Area}(U_r) = \frac{1}{\pi}\,\mathrm{Area}(U),$$

which completes the proof. □

3 Convexity of variance along rays

By Theorem 2, I_2 is a constant. As T is a non-degenerate triangle, $I_1 : T \to \mathbb{R}$ is a strictly positive function. It follows from (1. A) that to show the variance restricted to γ is a convex function, it is enough to show that I_1 restricted to γ is a concave function.

Our approach to proving Theorem 3 relies on the following special case where γ is an edge of the triangle.

Theorem 4. *Let T be a triangle with vertices $(0,0), (1,0)$ and (P,Q) where $P \in \mathbb{R}$ and $Q > 0$. Define $h : [0,1] \to \mathbb{R}$ by $h(x) = I_1(x,0)$. Then h is a concave function.*

With this in hand, the proof of Theorem 3 is as follows.

Proof of Theorem 3. Let T be a triangle with vertices u, v and w. Without loss of generality, suppose that $\gamma : [0,1] \to T$ parameterizes a line segment from the vertex u to a point u' on the side of T joining v to w. Then γ divides T into two triangles T_1 and T_2 with vertices u, u', v and u, u', w respectively.

Let f be a similarity map with scaling factor C which maps the line segment $[u, u']$ onto the line segment L joining $(0,0)$ to $(1,0)$. Then by the proof of Theorem 1,

$$\begin{aligned} I_1^T(z) &= \frac{1}{2\pi}\int_0^{2\pi} d_z^T(\theta)\, d\theta \\ &= \frac{1}{2\pi C}\int_0^{\pi} d_{f(z)}^{f(T)}(\theta)\, d\theta + \frac{1}{2\pi C}\int_\pi^{2\pi} d_{f(z)}^{f(T)}(\theta)\, d\theta \\ &= \frac{1}{C} I_1^{T_1}(f(z)) + \frac{1}{C} I_1^{T_2}(f(z)). \end{aligned}$$

By Theorem 4, $I_1^{T_1}|_L$ is concave and, by applying a reflection and again using Theorem 4, $I_1^{T_2}|_L$ is also concave. As the sum of two concave functions is concave, it follows that $I_1^T|_\gamma$ is a concave function. By the observation at the start of this section, it follows that the variance restricted to γ is a convex function, which completes the proof. □

The rest of this section is devoted to the proof of Theorem 4.

3.1 Computing I_1 In the following lemma, we explicitly compute the integral in the formula for I_1 from a vertex of a triangle.

Lemma 5. *Let T be a triangle with vertices u,v,w and side lengths opposite each vertex a,b,c respectively. Let p be the perpendicular distance from u to the line passing through v and w. Then*

$$I_1(u) = \frac{p}{2\pi}\left[\ln(a+b+c) - \ln(-a+b+c)\right]. \tag{3. B}$$

Proof. Let ϕ_u, ϕ_v and ϕ_w be the angles subtended at the vertices u, v and w respectively. We set $\theta = 0$ to correspond the the direction from u towards v and $\theta = \pi - \phi_v - \phi_w$ to be the direction from u towards w. Then it follows from elementary trigonometry that

$$\cos(\pi/2 - \phi_v - \theta) = \frac{p}{d_u(\theta)}.$$

Using the identity $\cos(\pi/2 - t) = \sin t$, we therefore have

$$d_u(\theta) = \frac{p}{\sin(\phi_v + \theta)}.$$

Now, by definition we have

$$I_1(u) = \frac{1}{2\pi}\int_0^{\pi-\phi_v-\phi_w} d_u(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi-\phi_v-\phi_w} p\csc(\phi_v+\theta)\,d\theta.$$

Making the change of variables $\alpha = \phi_v + \theta$, we obtain

$$\begin{aligned} I_1(u) &= \frac{1}{2\pi}\int_{\phi_v}^{\pi-\phi_w} p\csc(\alpha)\,d\alpha \\ &= \frac{1}{2\pi}\left[\ln\left|\frac{1-\cos\alpha}{\sin\alpha}\right|\right]_{\alpha=\phi_v}^{\pi-\phi_w} \\ &= \frac{1}{2\pi}\left[\ln\left|\tan\left(\frac{\alpha}{2}\right)\right|\right]_{\alpha=\phi_v}^{\pi-\phi_w} \\ &= \frac{p}{2\pi}\left(\ln\left|\tan\left(\frac{\pi-\phi_w}{2}\right)\right| - \ln\left|\tan\left(\frac{\phi_v}{2}\right)\right|\right] \\ &= \frac{p}{2\pi}\ln\left|\cot(\phi_v/2)\cot(\phi_w/2)\right|. \end{aligned}$$

Here we have used the half-angle formula for tan and the identity $\tan(\pi/2-t) = \cot t$. Let $s = (a+b+c)/2$ be the semi-perimeter of T. Then by the law of cotangents we have

$$\frac{\cot(\phi_v/2)}{s-b} = \frac{\cot(\phi_w/2)}{s-c} = \frac{1}{r},$$

where

$$r = \left(\frac{(s-a)(s-b)(s-c)}{s}\right)^{1/2}.$$

Combining this with the expression for $I_1(u)$ above, we find that

$$\begin{aligned} I_1(u) &= \frac{p}{2\pi}\ln\left|\left(\frac{s-b}{r}\right)\left(\frac{s-c}{r}\right)\right| \\ &= \frac{p}{2\pi}(\ln s - \ln(s-a)) \end{aligned}$$

which gives (3. B) once the $\ln(1/2)$ terms are cancelled. □

3.2 Splitting T into two sub-triangles

For $x \in (0,1)$, we decompose T into two triangles A, B where A has vertices $(0,0)$, $(x,0)$ and (P,Q) and B has vertices $(x,0)$, $(1,0)$ and (P,Q). Let $J_1 \in [0,2\pi)$ be the interval corresponding to angles giving triangle A viewed from x, and let J_2 be the corresponding interval for B. For $\theta \in [0,2\pi)\setminus(J_1 \cup J_2)$ we have $d_x(\theta) = 0$ and so

$$h(x) = \frac{1}{2\pi}\left(\int_{J_1} d_x(\theta)\,d\theta + \int_{J_2} d_x(\theta)\,d\theta\right).$$

For $i \in \{1,2\}$, let

$$f_i(x) = \int_{J_i} d_x(\theta)\,d\theta.$$

Here we are using a slight abuse of notation to write d_x instead of $d_{(x,0)}$.

Lemma 6. *We have*

$$f_1(x) = \frac{Qx}{\sqrt{P^2+Q^2}}\ln\left(\frac{\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}{-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}\right)$$

and

$$f_2(x) = \frac{Q(1-x)}{\sqrt{(P-1)^2+Q^2}}\ln\left(\frac{\sqrt{(P-1)^2+Q^2}+(1-x)+\sqrt{(P-x)^2+Q^2}}{-\sqrt{(P-1)^2+Q^2}+(1-x)+\sqrt{(P-x)^2+Q^2}}\right).$$

Proof. An elementary computation shows that the perpendicular distance from x to the line passing through $(0,0)$ and (P,Q) is Qx and the perpendicular distance from x to the line passing through $(1,0)$ and (P,Q) is $Q(1-x)$. The lemma then follows by applying Lemma 5 twice. □

Lemma 7. *We have*

$$f_1'(x) = \frac{Q}{\sqrt{P^2+Q^2}}\ln\left(\frac{\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}{-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}\right) - \frac{Q}{((P-x)^2+Q^2)^{1/2}} \tag{3. C}$$

and

$$f_1''(x) = Q((P-x)^2+Q^2)^{-3/2}\left(P - \frac{P^2+Q^2}{x}\right). \tag{3. D}$$

Proof. The derivative of $\ln(\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2})$ is

$$\frac{1-(P-x)/\sqrt{(P-x)^2+Q^2}}{\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}$$

and the derivative of $\ln(-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2})$ is

$$\frac{1-(P-x)/\sqrt{(P-x)^2+Q^2}}{-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}.$$

Thus

$$f_1'(x) = \frac{Q}{\sqrt{P^2+Q^2}} \ln\left(\frac{\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}{-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}\right)$$
$$+\frac{Qx}{\sqrt{P^2+Q^2}}\left(1-\frac{(P-x)}{\sqrt{(P-x)^2+Q^2}}\right)\left[\frac{1}{\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}-\right.$$
$$\left.\frac{1}{-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}\right].$$

By putting the terms over a common denominator, the term in the square brackets is

$$\frac{-2\sqrt{P^2+Q^2}}{-(P^2+Q^2)+(x+\sqrt{(P-x)^2+Q^2})^2} = \frac{-2\sqrt{P^2+Q^2}}{2x^2-2xP+2x\sqrt{(P-x)^2+Q^2}}$$
$$= \frac{-\sqrt{P^2+Q^2}}{x(x-P+\sqrt{(P-x)^2+Q^2})}.$$

We also note that

$$1-\frac{P-x}{\sqrt{(P-x)^2+Q^2}} = \frac{\sqrt{(P-x)^2+Q^2}+x-P}{\sqrt{(P-x)^2+Q^2}}$$

from which it follows that

$$f_1'(x) = \frac{Q}{\sqrt{P^2+Q^2}} \ln\left(\frac{\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}{-\sqrt{P^2+Q^2}+x+\sqrt{(P-x)^2+Q^2}}\right)$$
$$+\frac{Qx}{\sqrt{P^2+Q^2}}\left(\frac{\sqrt{(P-x)^2+Q^2}+x-P}{\sqrt{(P-x)^2+Q^2}}\right)\left[\frac{-\sqrt{P^2+Q^2}}{x(x-P+\sqrt{(P-x)^2+Q^2})}\right].$$

By cancelling terms here, we obtain (3. C). For the second derivative, we can use the work above to differentiate the first term in (3. C) and the quotient rule for the second term to obtain

$$\begin{aligned} f_1''(x) &= -\frac{Q}{x\sqrt{(P-x)^2+Q^2}} - \frac{Q(P-x)}{((P-x)^2+Q^2)^{3/2}} \\ &= Q\left(\frac{-(P-x)^2-Q^2-Px+x^2}{x((P-x)^2+Q^2)^{3/2}}\right) \\ &= -\frac{Q(P^2+Q^2-Px)}{x((P-x)^2+Q^2)^{3/2}} \end{aligned}$$

which is (3. D). □

Next we give the analogous result for f_2.

Lemma 8. *We have*

$$f_2'(x) = -\frac{Q}{\sqrt{(P-1)^2+Q^2}} \ln\left(\frac{\sqrt{(P-1)^2+Q^2}+(1-x)+\sqrt{(P-x)^2+Q^2}}{-\sqrt{(P-1)^2+Q^2}+(1-x)+\sqrt{(P-x)^2+Q^2}}\right)$$
$$+\frac{Q}{((P-x)^2+Q^2)^{1/2}} \qquad (3.\ E)$$

and

$$f_2''(x) = Q((P-x)^2+Q^2)^{-3/2}\left(1-P-\frac{(P-1)^2+Q^2}{1-x}\right). \tag{3. F}$$

Proof. The theme of the computation is similar to that of the proof of Lemma 7 and so we omit the details of the derivation of (3. E). Differentiating (3. E) yields

$$\begin{aligned} f_2''(x) &= -\frac{Q}{(1-x)\sqrt{(P-x)^2+Q^2}} + \frac{Q(P-x)}{((P-x)^2+Q^2)^{3/2}} \\ &= Q((P-x)^2+Q^2)^{-3/2}\left(-\left(\frac{(P-x)^2+Q^2}{1-x}\right)+P-x\right) \\ &= Q((P-x)^2+Q^2)^{-3/2}\left(-\left(\frac{(P-1+1-x)^2+Q^2}{1-x}\right)+P-1+1-x\right) \\ &= Q((P-x)^2+Q^2)^{-3/2}\left(1-P-\frac{(P-1)^2+Q^2}{1-x}\right) \end{aligned}$$

which is (3. F). □

By combining (3. D) and (3. F), we see that

$$\begin{aligned} h''(x) &= \frac{Q((P-x)^2+Q^2)^{-3/2}}{2\pi}\left(P-\frac{P^2+Q^2}{x}+1-P-\frac{(P-1)^2+Q^2}{1-x}\right) \\ &= \frac{Q((P-x)^2+Q^2)^{-3/2}}{2\pi}\left(1-\frac{P^2+Q^2}{x}-\frac{(P-1)^2+Q^2}{1-x}\right). \end{aligned} \tag{3. G}$$

For $x \in (0,1)$, set

$$j(x) = 1-\frac{P^2+Q^2}{x}-\frac{(P-1)^2+Q^2}{1-x}. \tag{3. H}$$

Then from (3. G), it is evident that h is a concave function if and only if $j(x)<0$ for $x\in(0,1)$.

3.3 Showing that j is always negative

Lemma 9. *We have* $\lim_{x\to 0^+} j(x) = -\infty$, $\lim_{x\to 1^-} j(x) = -\infty$, *and j has a unique critical point on* $(0,1)$ *at*

$$x_0 = \frac{\sqrt{P^2+Q^2}}{\sqrt{P^2+Q^2}+\sqrt{(P-1)^2+Q^2}} \tag{3. I}$$

with critical value

$$j(x_0) = 1-\left(P^2+Q^2+2\sqrt{P^2+Q^2}\sqrt{(P-1)^2+Q^2}+(P-1)^2+Q^2\right). \tag{3. J}$$

Moreover, for any $P\in\mathbb{R}$ and $Q>0$, we have $j(x_0)<0$.

Proof. The claims about the limits follow immediately from (3. H). Next,

$$j'(x) = \frac{P^2+Q^2}{x^2}-\frac{(P-1)^2+Q^2}{(1-x)^2}.$$

Solving $j'(x)=0$ yields

$$(1-x)^2(P^2+Q^2) = x^2((P-1)^2+Q^2).$$

As $x \in (0,1)$, we may take positive square roots of both sides to obtain

$$(1-x)\sqrt{P^2+Q^2} = x\sqrt{(P-1)^2+Q^2}$$

and solving for x yields (3. I). Plugging this value into the formula for j and some elementary algebra then yields (3. J).
For the final claim, set $J(P,Q)$ to be the right hand side of (3. J) as a function in terms of P and Q. Then

$$\begin{aligned} J(P,0) &= 1 - \left(P^2 + 2\sqrt{P^2}\sqrt{(P-1)^2} + (P-1)^2\right) \\ &= -2P(P-1) - 2|P(P-1)|. \end{aligned}$$

If $P(P-1) \le 0$, then $J(P,0) = 0$, otherwise $J(P,0) < 0$. In either case, we have $J(P,0) \le 0$. Now, for $Q > 0$, the partial derivative of J with respect to Q is

$$J_Q = -4Q - \frac{2Q\sqrt{P^2+Q^2}}{\sqrt{(P-1)^2+Q^2}} - \frac{2Q\sqrt{(P-1)^2+Q^2}}{\sqrt{P^2+Q^2}} < 0.$$

We conclude that for any fixed $P \in \mathbb{R}$, $\lim_{Q\to 0^+} J(P,Q) \le 0$ and $J_Q < 0$, from which it follows that $J(P,Q) < 0$ for all $P \in \mathbb{R}$ and $Q > 0$. This gives the final claim. □

From Lemma 9, we conclude that the unique critical point of j on $(0,1)$ is a maximum, and that maximum value is negative. Thus, by (3. G), h is a concave function on $[0,1]$ which completes the proof of Theorem 4.

4 Concluding remarks

The topic of this paper was a Research Rookies project undertaken by the third named author at NIU during the 2022-2023 academic year with the first named author as advisor. The authors would like to thank the organizers of the Research Rookies program at NIU for stimulating this work and thank the anonymous referee for comments which improved the exposition in the paper.
The images were created using Julia and Makie, see [1] and [2] respectively. We finish with another visual example of the variance. Note the different scales on the axes in Figures 3 and 4. This triangle is long and thin, but for ease of visual representation, the scales were chosen this way. Comparing Figures 1 and 2 with 3 and 4, we see that the variance achieves much higher values for long, thin triangles.

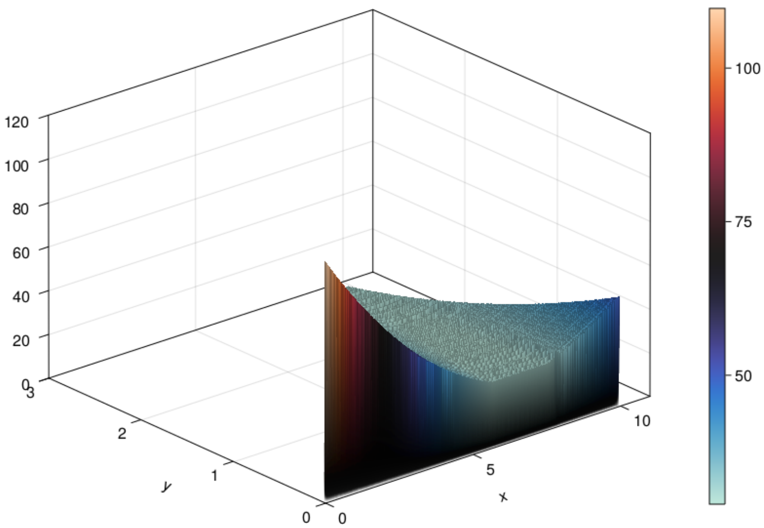


Figure 3: Surface plot generated with 99620 variance values with step size for x and y coordinates equal to 0.01.

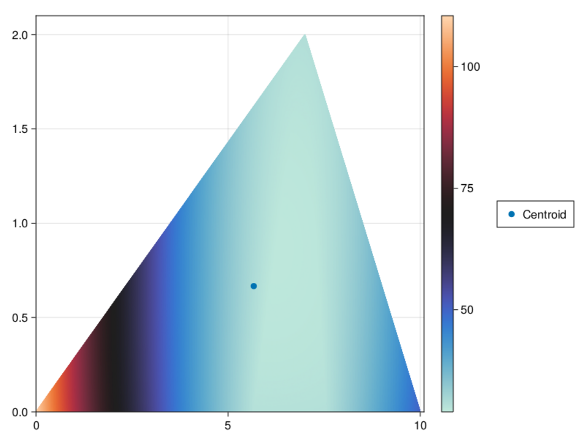


Figure 4: Flattened version of Figure 3 with the centroid marked.

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