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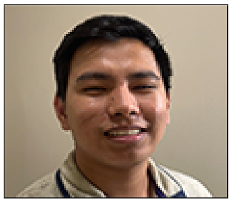
Continued Fractions, a-Fibonacci numbers, and the middle b-noise

*Aakash Gurung, Cheng-Han Pan**



Aakash Gurung is a junior at the University of Alabama, pursuing a dual major in Mathematics and Physics. This project was undertaken during his time at Juniata College. He is currently exploring diverse areas of mathematics, driven by a long-term aspiration to pursue a career in mathematical research.

Cheng-Han Pan received his Ph.D. in mathematics from West Virginia University. He served as a visiting assistant professor and faculty advisor for Juniata Problem Solving Group at Juniata College before joining Western New England University. His research interests focus on the foundations of real analysis, especially exploring paradoxical examples of functions and sets.

Abstract

Problem 1186 in The College Mathematics Journal asked for a closed form expression of the continued fraction $[1,1,\dots,1,3,1,1,\dots,1]$, and reappeared as Problem 1385 in the PME journal. In this paper, we present a generalization to $[a,a,\dots,a,b,a,a,\dots,a]$ with a-Fibonacci numbers and discuss how much the middle b-noise would impact the continued fractions with all a's.

1 Introduction

This paper is based on our participation in the Juniata Problem Solving Group in Fall 2022 and our further discoveries in one problem that we solved which can be traced to Problem 1186, proposed by Gregory Dresden and ZhenShu Luan, in The College Mathematics Journal [5]. The problem states: *Find a closed-form expression for the continued fraction* $[1,1,\dots,1,3,1,1,\dots,1]$, *which*

***Corresponding author**: cp621920@wne.edu

has n ones before and after, the middle three.

A simple finite continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}},$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}^+$ for $i \geq 1$. It is often denoted by $[a_0, a_1, a_2, \ldots, a_{n-1}, a_n]$. For example,

$$[1,3,1] = 1 + \frac{1}{3+\frac{1}{1}} = \frac{5}{4} \qquad \text{and} \qquad [1,1,3,1,1] = 1 + \cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\frac{1}{1}}}} = \frac{16}{9}.$$

Although a solution by Walther Janous was quickly published in [6], a closed-form expression can be derived differently. Since the Juniata Problem Solving Group did not exist until Fall 2022, the problem that later caught our attention was reproposed by Hongwei Chen as PME Problem 1385 in [3].

In PME Problem 1385, after a nice introduction to continued fractions and Fibonacci numbers, the proposers stated that they found a neat closed-form expression for $[1, 1, \ldots, 1, 3, 1, 1, \ldots, 1]$ and challenged the readers to prove it. In particular, readers are invited to show that

$$[\underbrace{1, 1, \ldots, 1}_{n\text{-times}}, 3, \underbrace{1, 1, \ldots, 1}_{n\text{-times}}] = \frac{F_{n+4}F_{n+1}}{F_{n+2}^2},$$

where F_n represents the nth Fibonacci number listed as follows. (fn. 1)

$\cdots\cdots$	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	$\cdots\cdots$
$\cdots\cdots$	1	0	1	1	2	3	5	8	13	21	34	$\cdots\cdots$

The Juniata Problem Solving Group submitted a solution to PME Problem 1385, (fn. 2) which was published in [4].

In the next section, we present a generalization from $[\underbrace{1, \ldots, 1}_{n\text{-times}}, 3, \underbrace{1, \ldots, 1}_{n\text{-times}}]$ to $[\underbrace{a, \ldots, a}_{n\text{-times}}, b, \underbrace{a, \ldots, a}_{n\text{-times}}]$ with a-Fibonacci numbers. In Section 3, we show how much the middle b changes the values of such a continued fraction when $b \neq a$ (the surprising answer is: not very much at all). Lastly in Section 4, we will come back and briefly address Juniata Problem Solving Group's approach in [4] and Walther Janous' approach in [3].

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1. Although Fibonacci numbers usually start with F_0 and F_1, note that one can extend the indexes of Fibonacci numbers to negative integers by using $F_n = F_{n+2} - F_{n+1}$. We will particularly need $F_{-1} = F_1 - F_0 = 1$ in some future formulas. On the other hand, Fibonacci numbers can be defined for negative indexes by $F_{-n} \coloneqq (-1)^{n+1}F_n$ as well.

2. Members included Emmanuel Adutwum, Aakash Gurung, Saea Eun Lee, and Johnathan Park at the time.

2 Generalization to $[a,\dots,a,b,a,\dots,a]$

Why just 1 and 3? An immediate generalization would be $[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]$ for $a,b \in \mathbb{Z}^+$. In this section, we will need to work with a-Fibonacci numbers. To increase the readability of the calculations in proofs throughout the paper, we will always use the shorthand notation $\mathbf{F}_n$ to represent the nth a-Fibonacci number.

For $a \in \mathbb{Z}^+$, the nth a-Fibonacci number, denoted by $\mathbf{F}_n$, is the nth term in the a-Fibonacci sequence defined recursively as

$$\mathbf{F}_n := \begin{cases} \mathbf{F}_{n+2} - a\mathbf{F}_{n+1} & \text{for } n \leq -1 \\ 0 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ a\mathbf{F}_{n-1} + \mathbf{F}_{n-2} & \text{for } n \geq 2 \end{cases}.$$

From the definition above, we see that a-Fibonacci numbers clearly generalize the original Fibonacci numbers. (fn. 3) Throughout the paper, we will repeatedly use the particular recursive relation $\mathbf{F}_n = a\mathbf{F}_{n-1} + \mathbf{F}_{n-2}$ to simplify our work. To acquaint readers with the recursive relation, we establish the validity of Cassini's identity, Honsberger's identity, and d'Ocagne's identity for a-Fibonacci numbers. These identities serve as essential tools in the proofs of Theorem 5, Corollary 7, and Remark 8.

Proposition 1. *(i) Cassini's identity: If $n \in \mathbb{N}$ and $a \in \mathbb{Z}^+$, then*

$$\boldsymbol{F}_{n+1}\boldsymbol{F}_{n-1} = (\boldsymbol{F}_n)^2 + (-1)^n.$$

(ii) Honsberger's identity: If $m,n \in \mathbb{N}$ and $a \in \mathbb{Z}^+$, then

$$\boldsymbol{F}_{m+n} = \boldsymbol{F}_m\boldsymbol{F}_{n+1} + \boldsymbol{F}_{m-1}\boldsymbol{F}_n.$$

(iii) d'Ocagne's identity: If $m,n \in \mathbb{N}$ and $a \in \mathbb{Z}^+$, then

$$(-1)^n\boldsymbol{F}_{m-n} = \boldsymbol{F}_m\boldsymbol{F}_{n+1} - \boldsymbol{F}_{m+1}\boldsymbol{F}_n.$$

Proof. To show (i), we run induction on $n \in \mathbb{N}$. It is easy to verify the basic step. $n = 0$ implies

$$\mathbf{F}_1\mathbf{F}_{-1} = 1 \cdot 1 = 1 = 0^2 + (-1)^0 = (\mathbf{F}_0)^2 + (-1)^0.$$

Suppose $\mathbf{F}_{k+1}\mathbf{F}_{k-1} = (\mathbf{F}_k)^2 + (-1)^k$ for some $k \geq 0$. We want to show that

$$\mathbf{F}_{k+2}\mathbf{F}_k = (\mathbf{F}_{k+1})^2 + (-1)^{k+1}.$$

Working on the left hand side, we break $\mathbf{F}_{k+2}$ into $a\mathbf{F}_{k+1} + \mathbf{F}_k$ and get

$$\mathbf{F}_{k+2}\mathbf{F}_k = (a\mathbf{F}_{k+1} + \mathbf{F}_k)\mathbf{F}_k = a\mathbf{F}_{k+1}\mathbf{F}_k + (\mathbf{F}_k)^2.$$

By the inductive hypothesis we replace $(\mathbf{F}_k)^2$ with $\mathbf{F}_{k+1}\mathbf{F}_{k-1} - (-1)^k$ and have

$$\begin{aligned} a\mathbf{F}_{k+1}\mathbf{F}_k + (\mathbf{F}_k)^2 &= a\mathbf{F}_{k+1}\mathbf{F}_k + \mathbf{F}_{k+1}\mathbf{F}_{k-1} - (-1)^k \\ &= \mathbf{F}_{k+1}(a\mathbf{F}_k + \mathbf{F}_{k-1}) - (-1)^k \\ &= \mathbf{F}_{k+1}(\mathbf{F}_{k+1}) - (-1)^k = (\mathbf{F}_{k+1})^2 + (-1)^{k+1} \end{aligned}$$

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3. Let $F_n(x)$ denote the nth Fibonacci Polynomial, which is another generalization of Fibonacci numbers. We want to mention that the nth a-Fibonacci number $\mathbf{F}_n$ can also be obtained by evaluating $F_n(a)$. This well-known fact can be easily proved by a strong induction.

as needed.
To see (ii), one would again run induction on $n \in \mathbb{N}$ while m is arbitrarily fixed. First, one shows the identity holds for $\mathbf{F}_{m+0}$ and $\mathbf{F}_{m+1}$. Assuming the identity holds for $\mathbf{F}_{m+k-1}$ and $\mathbf{F}_{m+k}$, the goal is to show that the identity also holds for $\mathbf{F}_{m+k+1} = a\mathbf{F}_{m+k} + \mathbf{F}_{m+k-1}$. We omit the straightforward but tedious details.
(iii) can be proved in the same way as described above. □

Lemma 2. *If $n \in \mathbb{N}$ and $a \in \mathbb{Z}^+$, then*

(i) $[\underbrace{a, a, \ldots, a}_{n\text{-times}}, x] = \dfrac{x\boldsymbol{F}_{n+1} + \boldsymbol{F}_n}{x\boldsymbol{F}_n + \boldsymbol{F}_{n-1}}$.

(ii) $[x, \underbrace{a, a, \ldots, a}_{n\text{-times}}] = x + \dfrac{\boldsymbol{F}_n}{\boldsymbol{F}_{n+1}}$.

Proof. For $n = 0$, $[x] = x = \frac{x \cdot 1 + 0}{x \cdot 0 + 1} = \frac{x\mathbf{F}_1 + \mathbf{F}_0}{x\mathbf{F}_0 + \mathbf{F}_{-1}}$ is clear. Suppose $[\underbrace{a, \ldots, a}_{k\text{-times}}, x] = \frac{x\mathbf{F}_{k+1} + \mathbf{F}_k}{x\mathbf{F}_k + \mathbf{F}_{k-1}}$ for some $k \geq 0$. We want to show that $[\underbrace{a, a, \ldots, a}_{(k+1)\text{-times}}, x] = \frac{x\mathbf{F}_{k+2} + \mathbf{F}_{k+1}}{x\mathbf{F}_{k+1} + \mathbf{F}_k}$. Indeed,

$$\begin{aligned}
[\underbrace{a, a, \ldots, a}_{(k+1)\text{-times}}, x] = a + \frac{1}{[\underbrace{a, \ldots, a}_{k\text{-times}}, x]} &= a + \frac{x\mathbf{F}_k + \mathbf{F}_{k-1}}{x\mathbf{F}_{k+1} + \mathbf{F}_k} \\
&= \frac{ax\mathbf{F}_{k+1} + a\mathbf{F}_k + x\mathbf{F}_k + \mathbf{F}_{k-1}}{x\mathbf{F}_{k+1} + \mathbf{F}_k} \\
&= \frac{x(a\mathbf{F}_{k+1} + \mathbf{F}_k) + a\mathbf{F}_k + \mathbf{F}_{k-1}}{x\mathbf{F}_{k+1} + \mathbf{F}_k} = \frac{x\mathbf{F}_{k+2} + \mathbf{F}_{k+1}}{x\mathbf{F}_{k+1} + \mathbf{F}_k}
\end{aligned}$$

completing the proof of (i). To see (ii), we simply apply (i) with $x = a$ and obtain

$$[x, \underbrace{a, a, \ldots, a}_{n\text{-times}}] = x + \frac{1}{[\underbrace{a, a, \ldots, a}_{n\text{-times}}]} = x + \frac{a\mathbf{F}_{n-1} + \mathbf{F}_{n-2}}{a\mathbf{F}_n + \mathbf{F}_{n-1}} = x + \frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}.$$

□

Moreover, the identity $[\underbrace{a, \ldots, a}_{n\text{-times}}] = \frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}$ is well-known and immediate from Lemma 2 with $x = a$. We are now able to derive a closed-form expression for $[\underbrace{a, \ldots, a}_{n\text{-times}}, b, \underbrace{a, \ldots, a}_{n\text{-times}}]$. Note that the expression in Lemma 3 may not be in its most elegant form, but it serves as the common ground for Theorem 5 and Remark 9.
Lemma 3. *If $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}^+$, then*

$$[\underbrace{a, \ldots, a}_{n\text{-times}}, b, \underbrace{a, \ldots, a}_{n\text{-times}}] = \frac{b(\boldsymbol{F}_{n+1})^2 + 2\boldsymbol{F}_{n+1}\boldsymbol{F}_n}{b\boldsymbol{F}_{n+1}\boldsymbol{F}_n + (\boldsymbol{F}_n)^2 + \boldsymbol{F}_{n+1}\boldsymbol{F}_{n-1}}.$$

Proof. We first write $[\underbrace{a, \ldots, a}_{n\text{-times}}, b, \underbrace{a, \ldots, a}_{n\text{-times}}]$ as $[\underbrace{a, \ldots, a}_{n\text{-times}}, [b, \underbrace{a, \ldots, a}_{n\text{-times}}]]$ so that Lemma

2 becomes applicable and gives

$$[\underbrace{a,\ldots,a}_{n\text{-times}},[b,\underbrace{a,\ldots,a}_{n\text{-times}}]]=[\underbrace{a,\ldots,a}_{n\text{-times}},b+\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}]=\frac{\left(b+\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}\right)\mathbf{F}_{n+1}+\mathbf{F}_n}{\left(b+\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}\right)\mathbf{F}_n+\mathbf{F}_{n-1}}.$$

After simplifying the complex fraction, we obtain

$$\begin{aligned}[\underbrace{a,\ldots,a}_{n\text{-times}},b,\underbrace{a,\ldots,a}_{n\text{-times}}]&=\frac{(b\mathbf{F}_{n+1}+\mathbf{F}_n)\mathbf{F}_{n+1}+\mathbf{F}_{n+1}\mathbf{F}_n}{(b\mathbf{F}_{n+1}+\mathbf{F}_n)\mathbf{F}_n+\mathbf{F}_{n+1}\mathbf{F}_{n-1}}\\&=\frac{b(\mathbf{F}_{n+1})^2+2\mathbf{F}_{n+1}\mathbf{F}_n}{b\mathbf{F}_{n+1}\mathbf{F}_n+(\mathbf{F}_n)^2+\mathbf{F}_{n+1}\mathbf{F}_{n-1}}\end{aligned}$$

as needed. □

3 The middle b-noise in convergents toward the ath metallic ratio

What do we mean by the middle b-noise? Comparing

$$[\underbrace{a,\ldots,a}_{n\text{-times}},b,\underbrace{a,\ldots,a}_{n\text{-times}}]\text{ and }[\underbrace{a,\ldots,a}_{n\text{-times}},a,\underbrace{a,\ldots,a}_{n\text{-times}}],$$

we wonder how much difference $b \neq a$ would cause. In this section, we will formulate the middle b-noise in different ways.

Remark 4. *It is well-known that the the infinite continued fraction* $[1,1,1,\ldots]$ *converges to the golden ratio. Moreover,* $[2,2,2,\ldots]$ *and* $[3,3,3,\ldots]$ *also converge, and those values are called the silver and the bronze ratios respectively. In fact, the infinite continued fraction* $[a,a,a,\ldots]$ *converges for each* $a \in \mathbb{Z}^+$*. In particular,*

$$[a,a,a,\cdots]=\frac{a+\sqrt{a^2+4}}{2},$$

and its value is called the ath metallic ratio.

Note that $[\underbrace{a,\ldots,a}_{n\text{-times}}]=\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}$ is called the nth convergent of $[a,a,a,\cdots]$. In Theorem 5 and Corollary 6, we first view

$$[\underbrace{a,\ldots,a}_{n\text{-times}},b,\underbrace{a,\ldots,a}_{n\text{-times}}]=[\underbrace{a,\ldots,a}_{n\text{-times}},[b,\underbrace{a,\ldots,a}_{n\text{-times}}]]$$

as $[\underbrace{a,\ldots,a}_{n\text{-times}}]$ attached with a $[b,\underbrace{a,\ldots,a}_{n\text{-times}}]$ tail part, and formulate how much the tail part alters the value of $[\underbrace{a,\ldots,a}_{n\text{-times}}]$.

Theorem 5. *If* $n \in \mathbb{N}$ *and* $a,b \in \mathbb{Z}^+$*, then*

$$[\underbrace{a,\ldots,a}_{n\text{-times}},b,\underbrace{a,\ldots,a}_{n\text{-times}}]=\frac{\boldsymbol{F}_{n+1}}{\boldsymbol{F}_n+\frac{(-1)^n}{b\boldsymbol{F}_{n+1}+2\boldsymbol{F}_n}}$$

Proof. To further rearrange the formula in Lemma 3, we simply apply Cassini's identity to replace $\mathbf{F}_{n+1}\mathbf{F}_{n-1}$ with $(\mathbf{F}_n)^2+(-1)^n$ and obtain

$$\begin{aligned}[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}] &= \frac{b(\mathbf{F}_{n+1})^2+2\mathbf{F}_{n+1}\mathbf{F}_n}{b\mathbf{F}_{n+1}\mathbf{F}_n+(\mathbf{F}_n)^2+(\mathbf{F}_n)^2+(-1)^n}\\ &= \frac{\mathbf{F}_{n+1}(b\mathbf{F}_{n+1}+2\mathbf{F}_n)}{\mathbf{F}_n(b\mathbf{F}_{n+1}+2\mathbf{F}_n)+(-1)^n} = \frac{\mathbf{F}_{n+1}}{\mathbf{F}_n+\frac{(-1)^n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}}\end{aligned}$$

as needed. ☐

From Theorem 5, It is easy to see that $[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]$ still converges to the ath metallic ratio as n goes to infinity. Indeed, the $[b,\underbrace{a,\dots,a}_{n\text{-times}}]$ tail part does not change the convergents in the long run since $\frac{(-1)^n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}$ tends to vanish as n goes to infinity.

How about the difference in value between $[\underbrace{a,\dots,a}_{n\text{-times}},[b,\underbrace{a,\dots,a}_{n\text{-times}}]]$ and $[\underbrace{a,\dots,a}_{n\text{-times}}]$ by subtraction? Instead of running into the unpleasant algebra of arranging a common denominator for

$$[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[\underbrace{a,\dots,a}_{n\text{-times}}] = \frac{\mathbf{F}_{n+1}}{\mathbf{F}_n+\frac{(-1)^n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}}-\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n},$$

we take the difference of their reciprocals since they already have the same denominator.

Corollary 6. *If $n\in\mathbb{N}$ and $a,b\in\mathbb{Z}^+$, then*

$$[0,\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[0,\underbrace{a,\dots,a}_{n\text{-times}}] = \frac{(-1)^n}{\boldsymbol{F}_{n+1}(b\boldsymbol{F}_{n+1}+2\boldsymbol{F}_n)}.$$

Proof.

$$\begin{aligned}[0,\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[0,\underbrace{a,\dots,a}_{n\text{-times}}] &= \frac{1}{[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]}-\frac{1}{[\underbrace{a,\dots,a}_{n\text{-times}}]}\\ &= \frac{\mathbf{F}_n+\frac{(-1)^n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}}{\mathbf{F}_{n+1}}-\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}} = \frac{(-1)^n}{\mathbf{F}_{n+1}(b\mathbf{F}_{n+1}+2\mathbf{F}_n)}.\end{aligned}$$

☐

Secondly, we view $[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]$ as $[\underbrace{a,\dots,a}_{n\text{-times}},a,\underbrace{a,\dots,a}_{n\text{-times}}]$ disturbed by the middle b-noise. Instead of comparing their actual difference, we again compare the difference of their reciprocals.

Corollary 7. *If $n\in\mathbb{N}$ and $a,b\in\mathbb{Z}^+$, then*

$$[0,\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[0,\underbrace{a,\dots\dots,a}_{(2n+1)\text{-times}}] = \frac{(-1)^n(a-b)}{(a\boldsymbol{F}_{n+1}+2\boldsymbol{F}_n)(b\boldsymbol{F}_{n+1}+2\boldsymbol{F}_n)}.$$

Proof. We prepare algebra rearrangements of $\frac{1}{[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]}$ and $\frac{1}{[\underbrace{a,\dots\dots,a}_{(2n+1)\text{-times}}]}$

separately before taking the subtraction. First, from Theorem 5 we have

$$\begin{aligned}\frac{1}{[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]} &= \frac{\mathbf{F}_n+\frac{(-1)^n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}}{\mathbf{F}_{n+1}} = \frac{(\mathbf{F}_n+\frac{(-1)^n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n})(\mathbf{F}_{n+2}+\mathbf{F}_n)}{\mathbf{F}_{n+1}(\mathbf{F}_{n+2}+\mathbf{F}_n)}\\ &= \frac{[\mathbf{F}_{n+2}\mathbf{F}_n+(\mathbf{F}_n)^2]+\frac{(-1)^n(\mathbf{F}_{n+2}+\mathbf{F}_n)}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}}{\mathbf{F}_{n+1}(\mathbf{F}_{n+2}+\mathbf{F}_n)}.\end{aligned}$$

Secondly, we want to rewrite $\frac{1}{[\underbrace{a,\dots\dots,a}_{(2n+1)\text{-times}}]} = \frac{\mathbf{F}_{2n+1}}{\mathbf{F}_{2n+2}}$. Applying Honsberger's identity on both $\mathbf{F}_{2n+1}$ and $\mathbf{F}_{2n+2}$ and using Cassini's identity to replace $(\mathbf{F}_{n+1})^2$ with $\mathbf{F}_{n+2}\mathbf{F}_n-(-1)^{n+1}$ give

$$\frac{\mathbf{F}_{2n+1}}{\mathbf{F}_{2n+2}} = \frac{\mathbf{F}_{(n+1)+n}}{\mathbf{F}_{(n+1)+(n+1)}} = \frac{(\mathbf{F}_{n+1})^2+(\mathbf{F}_n)^2}{\mathbf{F}_{n+1}\mathbf{F}_{n+2}+\mathbf{F}_n\mathbf{F}_{n+1}} = \frac{[\mathbf{F}_{n+2}\mathbf{F}_n+(\mathbf{F}_n)^2]-(-1)^{n+1}}{\mathbf{F}_{n+1}(\mathbf{F}_{n+2}+\mathbf{F}_n)}.$$

Notice that $[\mathbf{F}_{n+2}\mathbf{F}_n+(\mathbf{F}_n)^2]$ in their numerators directly cancel each other in the subtraction. We continue simplify their difference and obtain

$$\begin{aligned}\frac{\frac{(-1)^n(\mathbf{F}_{n+2}+\mathbf{F}_n)}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}+(-1)^{n+1}}{\mathbf{F}_{n+1}(\mathbf{F}_{n+2}+\mathbf{F}_n)} &= \frac{(-1)^n\left(\frac{a\mathbf{F}_{n+1}+2\mathbf{F}_n}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}-1\right)}{\mathbf{F}_{n+1}(a\mathbf{F}_{n+1}+2\mathbf{F}_n)} = \frac{(-1)^n\frac{(a-b)\mathbf{F}_{n+1}}{b\mathbf{F}_{n+1}+2\mathbf{F}_n}}{\mathbf{F}_{n+1}(a\mathbf{F}_{n+1}+2\mathbf{F}_n)}\\ &= \frac{(-1)^n(a-b)}{(a\mathbf{F}_{n+1}+2\mathbf{F}_n)(b\mathbf{F}_{n+1}+2\mathbf{F}_n)}\end{aligned}$$

as needed. □

Putting Corollary 6 and Corollary 7 together, not only did the similarity between the formulas interest us but also the role of the $a-b$ term in Corollary 7. If we consider $b\to\infty$, then we are no longer restricted to their reciprocals. We include our observations in the next remark.

Remark 8. *The difference between* $[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]$ *and* $[\underbrace{a,\dots,a}_{n\text{-times}}]$ *is independent of the distance between* a *and* b*. In fact, as* b *goes to infinity, the difference goes to* 0*. In particular, we have*

$$\begin{aligned}\lim_{b\to\infty}[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[\underbrace{a,\dots,a}_{n\text{-times}}] &= \lim_{b\to\infty}[\underbrace{a,\dots,a}_{n\text{-times}}+\frac{1}{b+X}]-[\underbrace{a,\dots,a}_{n\text{-times}}]\\ &= [\underbrace{a,\dots,a}_{n\text{-times}}]-[\underbrace{a,\dots,a}_{n\text{-times}}]=0,\end{aligned}$$

where $X=[0,\underbrace{a,\dots,a}_{n\text{-times}}]$*. Moreover, the limit is the same regardless the value of* X*.*

On the contrary, the difference between $[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]$ *and* $[\underbrace{a,\dots\dots,a}_{(2n+1)\text{-times}}]$ *is obviously dependent on the distance between* a *and* b*. Moreover, we have*

(i) $\lim_{b\to\infty}[\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[\underbrace{a,\dots\dots,a}_{(2n+1)\text{-times}}] = \frac{(-1)^n\boldsymbol{F}_{n+1}}{\boldsymbol{F}_{2n+1}\boldsymbol{F}_n}.$

(ii) $\lim\limits_{b\to\infty}[0,\underbrace{a,\dots,a}_{n\text{-}times},b,\underbrace{a,\dots,a}_{n\text{-}times}]-[0,\ \underbrace{a,\dots\dots,a}_{(2n+1)\text{-}times}\ ]=\dfrac{(-1)^{n+1}}{\mathbf{F}_{2n+2}}.$

Proof. Apply the limits. We are indeed computing $\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}-\frac{\mathbf{F}_{2n+2}}{\mathbf{F}_{2n+1}}$ and $\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}-\frac{\mathbf{F}_{2n+1}}{\mathbf{F}_{2n+2}}$ for (i) and (ii) respectively. From d'Ocagne's identity with $m=2n+1$, $\mathbf{F}_{2n+1}\mathbf{F}_{n+1}-\mathbf{F}_{2n+2}\mathbf{F}_n=(-1)^n\mathbf{F}_{n+1}$. We immediately have

$$\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}-\frac{\mathbf{F}_{2n+2}}{\mathbf{F}_{2n+1}}=\frac{\mathbf{F}_{2n+1}\mathbf{F}_{n+1}-\mathbf{F}_{2n+2}\mathbf{F}_n}{\mathbf{F}_{2n+1}\mathbf{F}_n}=\frac{(-1)^n\mathbf{F}_{n+1}}{\mathbf{F}_{2n+1}\mathbf{F}_n}$$

and

$$\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}-\frac{\mathbf{F}_{2n+1}}{\mathbf{F}_{2n+2}}=\frac{\mathbf{F}_{2n+2}\mathbf{F}_n-\mathbf{F}_{2n+1}\mathbf{F}_{n+1}}{\mathbf{F}_{2n+2}\mathbf{F}_{n+1}}=\frac{(-1)^{n+1}\mathbf{F}_{n+1}}{\mathbf{F}_{2n+2}\mathbf{F}_{n+1}}=\frac{(-1)^{n+1}}{\mathbf{F}_{2n+2}}$$

as needed. ☐

Alternately, both (i) and (ii) can be obtained without d'Ocagne's identity. Using Honsberger's identity, it is easy to see that $\mathbf{F}_{2n+2}=(a\mathbf{F}_{n+1}+2\mathbf{F}_n)\mathbf{F}_{n+1}$. (fn. 4) From Corollary 7, we have

$$\begin{aligned}\lim_{b\to\infty}[0,\underbrace{a,\dots,a}_{n\text{-times}},b,\underbrace{a,\dots,a}_{n\text{-times}}]-[0,\ \underbrace{a,\dots\dots,a}_{(2n+1)\text{-times}}\ ]&=\lim_{b\to\infty}\frac{(-1)^n(a-b)}{(a\mathbf{F}_{n+1}+2\mathbf{F}_n)(b\mathbf{F}_{n+1}+2\mathbf{F}_n)}\\&=\frac{(-1)^n(-1)}{(a\mathbf{F}_{n+1}+2\mathbf{F}_n)\mathbf{F}_{n+1}}=\frac{(-1)^{n+1}}{\mathbf{F}_{2n+2}},\end{aligned}$$

which is (ii). With (ii) and a simple move in algebra, we can obtain (i). (fn. 5) In the end of this section, we introduce some terminology connected to the first three metallic ratios. Recall that $[\underbrace{a,\dots,a}_{n\text{-times}}]=\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}$, where $\mathbf{F}_n$ denotes the nth a-Fibonacci number, converges to the golden, the silver, and the bronze ratio when $a=1,2,3$ respectively. As 1-Fibonacci numbers are the original ones, 2-Fibonacci and 3-Fibonacci numbers are also called the silver and the bronze Fibonacci numbers respectively. More interestingly for $n\in\mathbb{Z}^+$, the nth 2-Fibonacci number coincides with the nth Pell number. Pell numbers are named after John Pell, since they appear in the study of Pells equations $x^2-2y^2=\mp1$. (fn. 6) In particular, the odd terms of the sequence of ordered pairs defined recursively as

$$(x_n,y_n):=\begin{cases}(1,1) & \text{for } n=1\\(3,2) & \text{for } n=2\\2(x_{n-1},y_{n-1})+(x_{n-2},y_{n-2}) & \text{for } n\geq 2\end{cases}$$

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4. In particular, $\mathbf{F}_{2n+2}=\mathbf{F}_{n+1}\mathbf{F}_{n+2}+\mathbf{F}_n\mathbf{F}_{n+1}=(\mathbf{F}_{n+2}+\mathbf{F}_n)\mathbf{F}_{n+1}=(a\mathbf{F}_{n+1}+2\mathbf{F}_n)\mathbf{F}_{n+1}$.

5. $\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}-\frac{\mathbf{F}_{2n+2}}{\mathbf{F}_{2n+1}}=-\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}\frac{\mathbf{F}_{2n+2}}{\mathbf{F}_{2n+1}}\left(\frac{\mathbf{F}_n}{\mathbf{F}_{n+1}}-\frac{\mathbf{F}_{2n+1}}{\mathbf{F}_{2n+2}}\right)=-\frac{\mathbf{F}_{n+1}}{\mathbf{F}_n}\frac{\mathbf{F}_{2n+2}}{\mathbf{F}_{2n+1}}\frac{(-1)^{n+1}}{\mathbf{F}_{2n+2}}=\frac{(-1)^n\mathbf{F}_{n+1}}{\mathbf{F}_n\mathbf{F}_{2n+1}}$.

6. Unfortunately, Euler erroneously attributed another English mathematician Lord William V. Brouncker's work to John Pell, who had negligible contribution to the study.

are solutions to $x^2-2y^2=-1$, and those even terms are solutions to $x^2-2y^2=1$. Note that y_n defined above is called the nth Pell number, and it indeed matches the nth 2-Fibonacci number. (fn. 7) With the original Fibonacci numbers, Pell numbers, and the bronze Fibonacci numbers, we present some nice identities in the next remark. (fn. 8)

Remark 9. *Let F_n, P_n, B_n denote the nth Fibonacci number, Pell number, bronze Fibonacci number respectively. Then we have*

(i) $[\underbrace{1,\dots,1}_{n\text{-times}},3,\underbrace{1,\dots,1}_{n\text{-times}}]=\dfrac{F_{n+4}F_{n+1}}{(F_{n+2})^2}.$

(ii) $[\underbrace{2,\dots,2}_{n\text{-times}},4,\underbrace{2,\dots,2}_{n\text{-times}}]=\dfrac{2P_{n+2}P_{n+1}}{(P_{n+1}+P_n)^2}.$

(iii) $[\underbrace{3,\dots,3}_{n\text{-times}},5,\underbrace{3,\dots,3}_{n\text{-times}}]=1+\dfrac{4(B_{n+1})^2-(B_n)^2}{(B_{n+1}+B_n)^2}.$

Proof. (i) is already proved in [4, 6]. Note that in the computation of (ii) and (iii), we will be using a simple identity that $a\mathbf{F}_{n+1}\mathbf{F}_n+\mathbf{F}_{n+1}\mathbf{F}_{n-1}=(\mathbf{F}_{n+1})^2$ for $a=2$ and $a=3$ respectively.

To see (ii), we take the formula from Lemma 3 with $a=2$ and $b=4$. In the case of $a=2$, we will be using $\mathbf{F}_{n+2}=2\mathbf{F}_{n+1}+\mathbf{F}_n$ recursively. Recall that 2-Fibonacci numbers and Pell numbers coincide, that is, $\mathbf{F}_n=P_n$ when $a=2$. Consequently, we have

$$\begin{aligned}[\underbrace{2,\dots,2}_{n\text{-times}},4,\underbrace{2,\dots,2}_{n\text{-times}}]&=\frac{4(P_{n+1})^2+2P_{n+1}P_n}{4P_{n+1}P_n+(P_n)^2+P_{n+1}P_{n-1}}\\&=\frac{2P_{n+1}(2P_{n+1}+P_n)}{2P_{n+1}P_n+(P_n)^2+2P_{n+1}P_n+P_{n+1}P_{n-1}}\\&=\frac{2P_{n+1}P_{n+2}}{2P_{n+1}P_n+(P_n)^2+(P_{n+1})^2}=\frac{2P_{n+1}P_{n+2}}{(P_{n+1}+P_n)^2}\end{aligned}$$

as needed.

To see (iii), we take the formula from Lemma 3 with $a=3$ and $b=5$. In the case of $a=3$, we will be using $\mathbf{F}_{n+2}=3\mathbf{F}_{n+1}+\mathbf{F}_n$ recursively. Recall that a 3-Fibonacci number is also called the bronze Fibonacci number, that is, $\mathbf{F}_n=B_n$

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7. See [2, Chapter 2] for more details.

8. It is a remark because we cannot claim they are all our discoveries. In particular, we contributed a proof to (i) while the closed-form expression was already given in [3]. Similarly, we provide a proof for (ii), but the closed-form expression is suggested by our referees.

when $a = 3$. Consequently, we have

$$\begin{aligned}[\underbrace{3,\ldots,3}_{n\text{-times}},5,\underbrace{3,\ldots,3}_{n\text{-times}}] &= \frac{5(B_{n+1})^2 + 2B_{n+1}B_n}{5B_{n+1}B_n + (B_n)^2 + B_{n+1}B_{n-1}} \\ &= \frac{4(B_{n+1})^2 + (B_{n+1})^2 + 2B_{n+1}B_n}{2B_{n+1}B_n + (B_n)^2 + 3B_{n+1}B_n + B_{n+1}B_{n-1}} \\ &= \frac{4(B_{n+1})^2 + (B_{n+1} + B_n)^2 - (B_n)^2}{2B_{n+1}B_n + (B_n)^2 + (B_{n+1})^2} \\ &= \frac{4(B_{n+1})^2 + (B_{n+1} + B_n)^2 - (B_n)^2}{(B_{n+1} + B_n)^2} \\ &= 1 + \frac{4(B_{n+1})^2 - (B_n)^2}{(B_{n+1} + B_n)^2}\end{aligned}$$

as needed. □

4 Two approaches to the same closed-form expression

While $[\underbrace{1,\ldots,1}_{n\text{-times}},3,\underbrace{1,\ldots,1}_{n\text{-times}}] = \frac{F_{n+4}F_{n+1}}{(F_{n+2})^2}$ is the original inspiration for this paper, it is worthwhile to briefly address the key idea in two other approaches that we know of.

In [4], Juniata Problem Solving Group observed how a continued fraction is evaluated and noticed the computation went "flipping the bottom value" and "adding 1" repeatedly. We therefore defined a function $f(x) := 1 + \frac{1}{x}$ that represented the evaluating process. For example, $[1,1,1,x] = f(f(f(x))) = f^{(3)}(x)$. We showed $f^{(n)}(x) = \frac{xF_{n+1}+F_n}{xF_n+F_{n-1}}$ and used it to compute

$$[\underbrace{1,1,\ldots,1}_{n\text{-times}},3,\underbrace{1,1,\ldots,1}_{n\text{-times}}] = f^{(n)}(2 + f^{(n)}(1)).$$

On the other hand, Walther Janous in [6] used

$$[\underbrace{1,1,\ldots,1}_{n\text{-times}}] = \frac{F_{n+1}}{F_n} \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

to compute the entries in the first column of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$. It worked elegantly because of the connection between

$$[a_1,a_2,\ldots,a_n] = \frac{P_n}{Q_n} \text{ and } \prod_{i=1}^{n} \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{bmatrix}.$$

More information regarding the matrix representation of continued fractions and their convergents can be found in [1].

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