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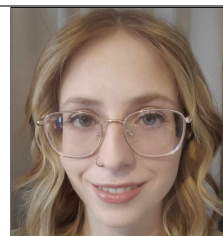
Catalan Number Sequences and Generalized Action Graphs

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1 Background

Action graphs emerged from work of Bergner and Hackney on category actions in the context of Reedy categories [4]. Alvarez, Bergner, and Lopez showed that action graphs could be inductively generated without reference to category actions, and they proved that the number of vertices added to A_n is the n -th Catalan number [2].

Definition 1.1 ([2]). The sequence of **action graphs** $\{A_n\}$ is defined inductively. Action graph A_0 is one vertex labeled 0 with no edges. Construct A_{k+1} from A_k by considering each vertex v in A_k . For each path from v to a vertex labeled k in A_k , add a new edge with source v to a new target vertex labeled $k+1$. Note that trivial paths are included in the paths considered from vertex v .

The first few graphs can be seen in Figure 1.

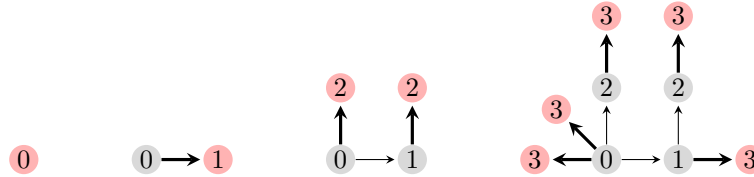


Figure 1: Action Graphs A_0 through A_3 , with new edges and vertices highlighted

Definition 1.2. The **Catalan numbers** are a sequence of natural numbers given by

$$C_0 = 1, \quad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} = \binom{2n}{n} \frac{1}{n+1}.$$

The first few numbers of the sequence are 1, 1, 2, 5, and 14.

The action graphs in [2] were further generalized in [3] by Cressman, Lin, Nguyen, and Wiljanen, who showed that the Fuss-Catalan numbers have a similar relation to another set of inductively defined directed graphs. The Fuss-Catalan numbers are a generalization of the Catalan numbers.

Definition 1.3 ([1], A14). The **Fuss-Catalan numbers** are defined by

$$C_{n,k} = \sum_{n_1+n_2+\dots+n_{k+1}=n-1} \prod_{i=1}^{k+1} C_{n_i,k} = \frac{\binom{n(k+1)}{n}}{kn+1}.$$

The sequence of the Fuss-Catalan numbers when $k=2$, that is $C_{n,2}$, is 1, 1, 3, 12, 55, ...

Observe that the Fuss-Catalan numbers agree with the Catalan numbers when $k=1$, that is $C_{n,1} = C_n$.

Cressman et al. expanded on the work of Alvarez, Bergner, and Lopez, developing new action graphs for the Fuss-Catalan numbers, which they called *generalized action graphs* [3].

Definition 1.4. [3] The **generalized action graphs** $T_{n,k}$ for the Fuss-Catalan numbers $C_{n,k}$ are defined inductively. For all k , $T_{0,k}$ is one vertex labeled 0 with no edges. Construct $T_{n,k}$ from $T_{n-1,k}$ by considering each vertex v in $T_{n-1,k}$. For each path of length ℓ from v to a vertex labeled $n-1$ in $T_{n-1,k}$, add $\binom{\ell+k-1}{\ell}$ new vertices labeled n and new edges from v to each of those new target vertices.

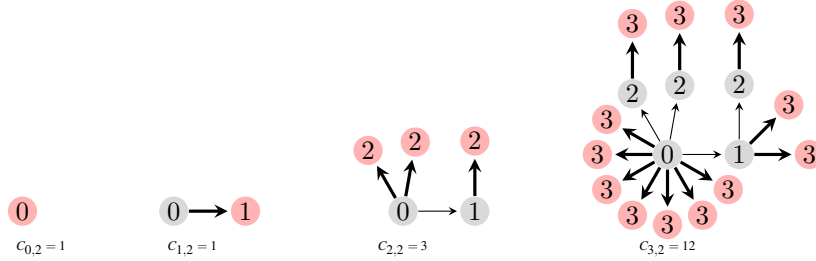


Figure 2: Generalized Action Graphs $T_{0,2}$ through $T_{3,2}$

Cressman et al. prove that the number of new vertices added to the generalized action graph $T_{n,k}$ is the Fuss-Catalan number $C_{n,k}$. In this paper, we first discuss our notion of generalized action graphs. Then, in Sections 2, 3, 4, and 5, we consider several other sequences related to the Catalan numbers, namely (a,b) -Catalan numbers, Catalan’s triangle, internal triangles, and super Catalan numbers respectively. We show action graphs cannot be generalized to Catalan’s triangle, (a,b) -Catalan numbers, nor internal triangles. We also conjecture a method for constructing action graphs for the super Catalan numbers.

1.1 Conventions and notation.

We use the following conventions for descendent and subtree in this paper.

Definition 1.5. A **descendent** of a vertex v in a directed graph is any vertex w such that there is a directed path from v to w . For a rooted tree T and a vertex v of T , the **subtree of T with root v** is the induced subgraph of vertex v and its descendants.

Because generalized action graphs can easily become unwieldy in size, we introduce the following condensed notation to ease bookkeeping.

Since generalized action graphs have many identical subgraphs with the same labels, we collapse them. For an edge from a vertex labeled k to a vertex labeled n , the multiplier, $\times m$, indicates the number of such edges from a vertex labeled k to vertices labeled n in the original graph. For a vertex in the condensed form, we can find the number of vertices it represents in the standard form by multiplying the labels along the path from the root to that vertex. For example, the upper right vertex labeled 2 in the condensed graph in Figure 3 represents the $2 \times 2 = 4$ vertices labeled 2 that come off of the two vertices labeled 1 in the original graph.

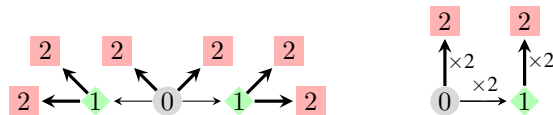


Figure 3: A directed rooted tree in standard and condensed form

As in Figure 3, we will sometimes change the color and shape of vertices in examples. This is intended only to highlight existing structure in the graphs, not to indicate any additional graph coloring or labeling. This paper will reference the sequences given in Table 1.

Name	Notation & Definition	Number Sequence
Catalan numbers	$C_n = \binom{2n}{n} \frac{1}{n+1}$	1, 1, 2, 5, 14, ...
Fuss-Catalan numbers	$C_{n,k} = \frac{1}{kn+1} \binom{n(k+1)}{n}$	$k = 1$: 1, 1, 2, 5, 14, ... $k = 2$: 1, 1, 3, 12, 55, ...
Catalan's triangle	$C(n, k) = \frac{n-k+1}{n+1} \binom{n+k}{k}$	Table 2
(a, b) -Catalan numbers	$Cat(a, b) = \frac{1}{a+b} \cdot \binom{a+b}{a}$	Table 3
Weak (a, b) -Catalan numbers	$cat(a, b) = \frac{1}{a+b} \cdot \binom{a+b}{a}$	Definition 2.2
Internal triangles	$t(n) = (n+2)C_{n-1} - 2C_n$	2, 14, 72, ...
Super Catalan numbers	$S(m, n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$	$m = 0$: 1, 2, 6, 20, 70, ... $m = 1$: 2, 2, 4, 10, 28, ...

Table 1: A table of sequences referenced in this paper.

1.2 Generalized action graph properties

We expand on the work of [3] by investigating whether similar graphs can be constructed for other sequences. Action graphs corresponding to the Catalan numbers and the Fuss-Catalan numbers have been defined, but there does not exist a general solution for any sequence of numbers in the literature. Based on the previous work on this topic, we define three axioms that describe generalized action graphs and explore some of the properties of such graphs.

Definition 1.6. The sequence $\{G_n\}$ of **generalized action graphs** for a particular sequence $\{s_n\}$ of positive integers is a sequence of directed, labeled graphs such that:

Axiom 1 We define G_0 as the graph with s_0 vertices labeled 0 and no edges.

We construct G_n from G_{n-1} by adding s_n new vertices, which are each labeled n .

Axiom 2 For vertex v in G_n , the subtree of G_n with root v is isomorphic to some G_k such that $k \leq n$.

Axiom 3 All leaves in the graph G_n have label n .

Based on these axioms, some necessary conditions for generalized action graphs follow.

Lemma 1. *In order for a sequence $\{s_n\}_{n \geq 0}$ to form a valid sequence of generalized action graphs, it must have the property that $s_0 = 1$.*

Proof. Assume that $\{s_n\}$ has the property that $s_0 \neq 1$. Then, G_0 would have multiple nodes labeled 0. If we consider the subtree rooted at any one of those 0 nodes in G_0 , that subtree will contain just a single node, and therefore is not

isomorphic to a generalized action graph in the sequence, which violates Axiom 2. Therefore, any sequence that has $s_0 \neq 1$ cannot have valid generalized action graphs. \square

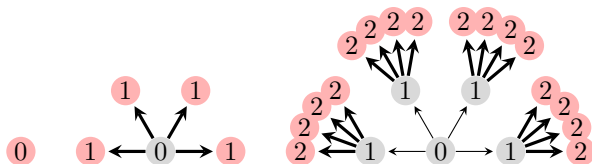
Lemma 2. *In order for a sequence $\{s_n\}_{n \geq 0}$ to form a valid sequence of generalized action graphs, it must satisfy $s_2 \geq s_1^2$.*

Proof. By Lemma 1, $s_0 = 1$, and thus G_0 is a single node labeled 0. Suppose for contradiction that $s_1 = a$ and $s_2 = b$ such that $a^2 > b$. To create G_1 , we add s_1 new leaves labeled 1. So G_1 has a total of a new nodes labeled 1 with edges from the single node labeled 0. Axiom 2 implies that in all future generalized action graphs, each node with path length zero to the newest node will result in a new nodes.

To create G_2 , we will add s_2 nodes labeled 2. However, as stated above, each node labeled 1 will result in a new nodes labeled 2. This forces a^2 nodes labeled 2 to be added to the graph, which is greater than our allowed $s_2 = b$ nodes labeled 2. Therefore, in order for a generalized action graph to exist, it must be that $s_2 \geq s_1^2$. \square

The following example gives a more visual representation of the property described by Lemma 2.

Example 3. *Consider a sequence $\{a_n\}$ given by $1, 4, 14, 48, \dots$. Since $a_0 = 1$, this creates a single node on the first step labeled as 0. Then, since $a_1 = 4$, four nodes labeled with 1 have edges from the node labeled 0, which sets the precedent (by Axiom 2) that new nodes get four new leaves in the next step. Then, in the third step, since the most recently added nodes are known to receive four new nodes as descendants, we conclude that there should be at least 16 nodes labeled 2. However, $a_2 = 14$. If we add 16 nodes, then the graph will not satisfy Axiom 1.*



2 (a, b) -Catalan numbers

A sequence of numbers that is closely related the Catalan numbers is the sequence of (a, b) -Catalan numbers.

Definition 2.1. [1, A16] For relatively prime integers $a \geq 0$ and $b \geq 0$, the (a, b) -Catalan number, denoted $Cat(a, b)$, is given by:

$$Cat(a, b) = \frac{1}{a+b} \cdot \binom{a+b}{a} = \frac{(a+b-1)!}{a!b!}.$$

Notice that the definition of an (a, b) -Catalan number requires that a and b be relatively prime integers. In order to assist in some of our exploration

and calculations, it is useful to drop the relatively prime requirement. For non-negative integers a and b , we define a generalization of the (a, b) -Catalan numbers that we call *weak (a, b) -Catalan numbers*. As far as we are aware, this sequence has not previously been studied.

Definition 2.2. For any non-negative integers a and b that are not both zero, the **weak (a, b) -Catalan number** is given by

$$\text{cat}(a, b) = \frac{1}{a+b} \cdot \binom{a+b}{a}.$$

Remark 4. An important difference between standard (a, b) -Catalan numbers, $\text{Cat}(a, b)$, and weak (a, b) -Catalan numbers, $\text{cat}(a, b)$, is that a standard (a, b) -Catalan number will always be an integer, while a weak (a, b) -Catalan number may not be an integer.

Example 5. Notice that $a = 3$ and $b = 6$ are not relatively prime numbers. We compute the weak (a, b) -Catalan number

$$\text{cat}(3, 6) = \frac{(3+6-1)!}{3!6!} = \frac{8!}{3!6!} = \frac{28}{3}.$$

For the remainder of this paper, when we refer to an (a, b) -Catalan number, we will mean the more general weak (a, b) -Catalan number.

2.1 Properties of (a, b) -Catalan numbers

We will discuss some important properties of (a, b) -Catalan numbers, including how they are connected to the Fuss-Catalan and Catalan numbers.

Observe that for any a and b , $\text{cat}(a, b) = \text{cat}(b, a)$ since

$$\text{cat}(a, b) = \frac{(a+b-1)!}{a!b!} = \frac{(b+a-1)!}{b!a!} = \text{cat}(b, a). \quad (2. A)$$

Next, we will see the connection between the Fuss-Catalan numbers and the (a, b) -Catalan Numbers. Note that $\text{cat}(n, kn+1) = \text{Cat}(n, kn+1)$ here because n and $kn+1$ are relatively prime.

Theorem 6. [1, A16a] For all $n \geq 0$ and $k \in \mathbb{N}$,

$$\text{cat}(n, kn+1) = C_{n,k}.$$

Proof. Recall that $C_{n,k} = \frac{1}{(kn+1)} \cdot \binom{n(k+1)}{n}$. Then, consider that

$$C_{n,k} = \frac{1}{(kn+1)} \binom{n(k+1)}{n} = \frac{1}{kn+1} \cdot \frac{(n(k+1))!}{n! \cdot (n(k+1)-n)!} = \frac{(nk+n)!}{n! \cdot (nk+1)!}.$$

Now, notice that

$$\text{cat}(n, kn+1) = \frac{(n+(kn+1)-1)!}{n!(kn+1)!} = \frac{(nk+n)!}{n!(nk+1)!}.$$

Thus, $\text{cat}(n, kn+1) = C_{n,k}$ for all $n \geq 0$ and $k \in \mathbb{N}$. \square

Corollary 7 ties the (a, b) -Catalan numbers in with the Catalan numbers. This result can be observed using the previous theorem and the connection between the Fuss-Catalan numbers and Catalan numbers, $C_{n,1} = C_n$.

Corollary 7. [1, A16a] For all $n \geq 0$,

$$cat(n, n+1) = C_n.$$

To assist in proving Proposition 9, we state the following lemma.

Lemma 8. For $a \in \mathbb{N}$, $cat(a, 1) = cat(1, a) = 1$.

Proof. Let $a \in \mathbb{N}$. By definition of (a, b) -Catalan number:

$$cat(a, 1) = \frac{(a+1-1)!}{a! \cdot 1!} = \frac{a!}{a!} = 1.$$

Applying Equation (2. A), $cat(a, 1) = cat(1, a) = 1$. \square

Our goal is to explore whether we can make generalized action graphs for the (a, b) -Catalan numbers. The following proposition will assist us in that exploration.

Proposition 9. For all $n \geq 4$,

$$n^2 \cdot cat(n+1, 1)^2 > n \cdot cat(n+2, 2).$$

Proof. We will prove this proposition using induction.

For the base case, consider $n = 4$. We have $4^2 cat(5, 1)^2 > 4 cat(6, 2)$ since $4^2 cat(5, 1)^2 = 16$ and $4 cat(6, 2) = 14$.

Suppose there is some $k \geq 4$ such that $k^2 \cdot cat(k+1, 1)^2 > k \cdot cat(k+2, 2)$. By Lemma 8, we know that $cat(a, 1) = 1$ for any $a \in \mathbb{N}$. So, we have

$$k^2 > k \cdot cat(k+2, 2).$$

Since $k > 0$, the above equation gives $k > cat(k+2, 2)$, which implies that $k \cdot \frac{k+4}{k+3} > cat(k+2, 2) \cdot \frac{k+4}{k+3}$. Note that by definition:

$$cat(k+2, 2) \cdot \frac{k+4}{k+3} = \frac{(k+3)!}{(k+2)!2!} \cdot \frac{k+4}{k+3} = \frac{(k+4)!}{(k+3)!2!} = cat(k+3, 2).$$

So, we have $k \cdot \frac{k+4}{k+3} > cat(k+3, 2)$.

Note that $m^2 + 4m + 3 > m^2 + 4m$ for any $m \in \mathbb{N}$. Since

$$(m+1)(m+3) = m^2 + 4m + 3 > m^2 + 4m = m(m+4),$$

we can conclude that $m+1 > m \cdot \frac{m+4}{m+3}$. Thus the same is true for k , that is

$$k+1 > k \cdot \frac{k+4}{k+3} > cat(k+3, 2).$$

Since $cat(k+2, 1)^2 = 1^2 = 1$ by Lemma 8, we know

$$k+1 = (k+1) \cdot cat(k+2, 1)^2,$$

and therefore $(k+1) \cdot cat(k+2, 1)^2 > cat(k+3, 2)$, which implies

$$(k+1)^2 \cdot cat(k+2, 1)^2 > (k+1) \cdot cat(k+3, 2).$$

This finishes the inductive step. \square

The following two lemmas provide other properties of (a, b) -Catalan numbers.

Lemma 10. For any non-negative integers a and b ,

$$cat(a, b) = cat(a, b-1) + \frac{a-1}{a} cat(a-1, b).$$

Proof. We can use the definition of the (a, b) -Catalan numbers and simplify:

$$\begin{aligned}
 cat(a, b-1) + \frac{a-1}{a} cat(a-1, b) &= \frac{(a+b-2)!}{a!(b-1)!} + \frac{a-1}{a} \frac{(a+b-2)!}{(a-1)!b!} \\
 &= \frac{b(a+b-2)!}{a!b!} + \frac{(a-1)(a+b-2)!}{a!b!} \\
 &= \frac{(a+b-2)!(b+(a-1))}{a!b!} \\
 &= \frac{(a+b-1)!}{a!b!} \\
 &= \frac{1}{a+b} \frac{(a+b)!}{a!b!} \\
 &= cat(a, b).
 \end{aligned}$$

□

Lemma 11. For any integer $n \geq 2$,

$$cat(3, n) = cat(3, n-1) + \frac{n+1}{3}.$$

Proof. We will proceed by induction. As a base case, consider $n = 2$. We have

$$cat(3, 2) = 2 = 1 + \frac{2+1}{3} = cat(3, 2-1) + \frac{2+1}{3}.$$

Assume for some $k \geq 2$ that $cat(3, k) = cat(3, k-1) + \frac{k+1}{3}$. Then consider,

$$\begin{aligned}
 cat(3, k+1) &= \frac{1}{3+k+1} \frac{(3+k+1)!}{3!(k+1)!} && \text{(def of } (a, b)\text{-Catalan)} \\
 &= \frac{k^2+5k+6}{3!} \\
 &= \frac{(k+1)k}{3!} + \frac{4k+6}{3!} \\
 &= \frac{(k+1)(k)}{3!} + \frac{2k+3}{3} \\
 &= \frac{1}{k+2} \frac{(k+2)!}{3!(k-1)!} + \frac{2k+3}{3} \\
 &= \frac{1}{3+k-1} \frac{(3+k-1)!}{3!(k-1)!} + \frac{2k+3}{3} \\
 &= cat(3, k-1) + \frac{k+1}{3} + \frac{k+2}{3} && \text{(def of } (a, b)\text{-Catalan)} \\
 &= cat(3, k) + \frac{k+2}{3} && \text{(inductive hypothesis)} \\
 &= cat(3, (k+1)-1) + \frac{(k+1)+1}{3}.
 \end{aligned}$$

Thus the inductive step holds, and our desired result follows. □

Observe that for arbitrary integers a and b , the weak (a, b) -Catalan number $cat(a, b)$ is generally a fraction. However, action graphs are constructed based

on whole number sequences. Therefore, we will introduce Catalan’s triangle as a sequence that is closely related to the (a, b) -Catalan numbers, as described in Lemma 12, but whose entries are whole numbers.

3 Catalan’s triangle

Another construction of numbers related to the Catalan numbers is Catalan’s triangle.

Definition 3.1 ([6]). **Catalan’s triangle** has entries denoted $C(n, k)$ that are defined as follows for integers n and k with $n \geq k \geq 0$:

$$C(n, k) = \frac{n - k + 1}{n + 1} \binom{n + k}{k}.$$

We compute the entries of Catalan’s triangle up to $n = 8$ in Table 2.

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	5	5					
4	1	4	9	14	14				
5	1	5	14	28	42	42			
6	1	6	20	48	90	132	132		
7	1	7	27	75	165	297	429	429	
8	1	8	35	110	275	572	1001	1430	1430

Table 2: A table of Catalan’s triangle

We make the following observations about the entries of Catalan’s triangle:

- (i) $C(n, 0) = 1$, for $n \geq 0$.
- (ii) $C(n, 1) = n$, for $n \geq 1$.
- (iii) $C(n + 1, k) = C(n + 1, k - 1) + C(n, k)$, for $1 < k < n + 1$.
- (iv) $C(n + 1, n + 1) = C(n + 1, n)$, for $n \geq 1$.

Because Catalan’s triangle is an array rather than a sequence, there are many ways to create sequences from the entries. We consider building generalized action graphs for the sequences that arise as columns, rows, and diagonals of the triangle. We will describe these results in Section 3.2, after we have shown the relationship between Catalan’s triangle and the (a, b) -Catalan numbers.

3.1 Relation to (a, b) -Catalan numbers

In this section, we will show that there is an interesting connection between the (a, b) -Catalan numbers and Catalan's triangle.

Lemma 12. *If $a \geq b \geq 0$ and $a \geq 1$, then $C(a-1, b) = (a-b)cat(a, b)$.*

Proof. Consider that for a Catalan's triangle entry $C(a-1, b)$,

$$\begin{aligned} C(a-1, b) &= \frac{a-b}{a} \binom{a+b-1}{b} \\ &= \frac{a-b}{a} \cdot \frac{(a+b-1)!}{b!(a-1)!} \\ &= \frac{(a-b)(a+b-1)!}{b!a!} \\ &= (a-b)cat(a, b). \end{aligned}$$

□

This leads to some methods for concluding whether generalized action graphs can be formed for these sequences. First, we look at a consequence of Lemma 12.

Lemma 13. *Any (a, b) -Catalan number with $a \neq b$ can be written as a fraction with denominator $|a-b|$. In particular, when $a > b$,*

$$cat(a, b) = \frac{C(a-1, b)}{a-b}.$$

Proof. Recall from Equation (2. A) that $cat(a, b) = cat(b, a)$. Without loss of generality, let $a > b$. By Lemma 12, $(a-b) \cdot cat(a, b) = C(a-1, b)$. Then $cat(a, b) = \frac{C(a-1, b)}{a-b}$, where $C(a-1, b)$ and $a-b$ are whole numbers. □

To demonstrate this re-writing of (a, b) -Catalan numbers, we have Table 3. Note that the diagonals represent where the differences in a and b are the same, and so they are written with the same denominators.

As previously noted, the weak (a, b) -Catalan numbers are frequently fractions rather than whole numbers. It is unclear to us what a fractional number of vertices should mean, so we will not try to directly construct generalized action graphs for this sequence. Instead, we will focus on the numerators in Table 3, which, as a result of the preceding lemmas, are Catalan's triangle numbers. Observe that the numerators of Table 3 are exactly the entries in Table 2.

3.2 Generalized action graphs for (a, b) -Catalan numbers and Catalan's triangle

In this section we will find that generalized action graphs for the most part cannot be made for sequences built from Catalan's triangle, and equivalently the numerators of the (a, b) -Catalan numbers. We will explore the columns, rows, and diagonals of Catalan's triangle.

First, we will consider the columns.

$a \backslash b$	0	1	2	3	4	5	6	7	8
1	1/1								
2	1/2	1/1							
3	1/3	2/2	2/1						
4	1/4	3/3	5/2	5/1					
5	1/5	4/4	9/3	14/2	14/1				
6	1/6	5/5	14/4	28/3	42/2	42/1			
7	1/7	6/6	20/5	48/4	90/3	132/2	132/1		
8	1/8	7/7	27/6	75/5	165/4	297/3	429/2	429/1	
9	1/9	8/8	35/7	110/6	275/5	572/4	1001/3	1430/2	1430/1

Table 3: (a, b) –Catalan Numbers

Proposition 14. *The columns of Catalan’s triangle do not correspond to generalized action graphs, except in the case where $k = 0$.*

Proof. The entries of the columns correspond to $C(n, k)$, where k is fixed and n is iterated. Except for the columns of $k = 0$ and $k = 1$, the columns have first entry greater than 1, since the $n = k$ diagonal is the Catalan numbers. (For illustration, see Table 2.) By Lemma 1, these columns cannot be used to make a generalized action graph. For the column where $k = 1$, notice that it contradicts Lemma 2, and thus we cannot make a generalized action graph with this column. Therefore, the sequences made from the columns of Catalan’s triangle cannot be modeled using (nontrivial) action graphs.

For the column where $k = 0$, we can make a somewhat trivial generalized action graph, where only one node is added each step. \square

Now, we will explore the rows.

Proposition 15. *The rows of Catalan’s triangle do not correspond to generalized action graphs.*

Proof. The entries of the rows correspond to $C(n, k)$, where n is fixed and k is iterated. First, we claim that for nonnegative n , $C(n, 2) < C(n, 1)^2$. To see this result, observe

$$C(n, 2) = \frac{n-2+1}{n+1} \binom{n+2}{2} = \frac{n-1}{n+1} \cdot \frac{(n+2)!}{2!((n+2)-2)!} = \frac{n^2+n-2}{2}.$$

Then by properties of Catalan’s triangle, $C(n, 1) = n$, thus $C(n, 1)^2 = n^2$. Now, consider that

$$\frac{n^2+n-2}{2} - n^2 = \frac{-n^2+n-2}{2}$$

and $\frac{-n^2+n-2}{2} < 0$ for all n . This implies that

$$\frac{n^2+n-2}{2} - n^2 = C(n,2) - C(n,1)^2 < 0,$$

and thus $C(n,2) < C(n,1)^2$.

So this means that using the rows to try and formulate generalized action graphs will lead to a contradiction of Lemma 2, since the second value, $C(n,1)$, when squared is always more than the third value in the sequence, $C(n,2)$. \square

Next, we will explore the diagonals. We will consider the i -th diagonal to be the sequence of entries $C(i+k, k)$ where k is iterated.

Example 16. *The third diagonal in Catalan's triangle is the sequence $C(3,0), C(4,1), C(5,2), C(6,3), \dots$*

Proposition 17. *Action graphs cannot be formulated for the i -th diagonal of Catalan's triangle when $i \geq 3$.*

Proof. Consider Proposition 9, which says that

$$m^2 \cdot \text{cat}(m+1, 1)^2 > m \cdot \text{cat}(m+2, 2)$$

for all $m \geq 4$. Combining this with Lemma 12, which implies that

$$\text{cat}(m+j, j) = \frac{1}{m} C(m+j-1, j),$$

we see that

$$m^2 \frac{1}{m^2} C(m+1-1, 1)^2 > m \frac{1}{m} C(m+2-1, 2),$$

and hence

$$C((m-1)+1, 1)^2 > C((m-1)+2, 2),$$

for $m-1 \geq 3$. This results in a contradiction with Lemma 2, which implies that we cannot construct generalized action graphs for the i -th diagonal of Catalan's triangle for any $i \geq 3$. \square

There still remain two Catalan's triangle diagonals to consider. The first diagonal consists of the Catalan numbers, and thus we know they have corresponding action graphs. Constructing generalized action graphs, or proving they do not exist, for the second diagonal of Catalan's triangle remains as future work.

4 Internal Triangles.

One application of the Catalan numbers is counting the number of triangulations of polygons, using diagonals that do not intersect in the interior of the polygon. More specifically, for a convex $(n+2)$ -gon \mathcal{P}_{n+1} , the number of triangulations is given by C_n [1]. There is a sequence related to these triangulations referred to as the *internal triangles*, which count the number of ways to draw triangles within a polygon without using the sides of the polygon.

Lemma 18 ([1], A5). *The total number of internal triangles of a polygon with $n + 2$ sides is given by*

$$t(n) = (n + 2)C_{n-1} - 2C_n = 2 \binom{2n-3}{n-4}.$$

Via either of these formulas, we can compute $t(4) = 2, t(5) = 14, t(6) = 72$, and so on.

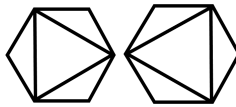


Figure 4: $t(4) = 2$, the number of internal triangles for hexagons.

We will use the term *path length rule* to refer to the number of new vertices added to a vertex based on the length of paths from the vertex to a leaf labeled n . For $n \geq 0$, ℓ_n denotes the number of new vertices added for each path of length n .

Theorem 19. *The internal triangles cannot be modeled using generalized action graphs.*

Proof. We will use path length rules to show that we cannot build consistent generalized action graphs beyond $t(6)$. Assume for a contradiction that we can model the internal triangle numbers using generalized action graphs. To satisfy Lemma 1, we begin with one vertex labeled 0. We must construct the subsequent graphs to satisfy the three generalized action graph axioms. Since $t(4) = 2$, there is a single trivial path from the zero vertex to itself and we must have two new vertices labeled 1 coming off of our 0 vertex. So, $\ell_0 = 2$. In the next generalized action graph, our ℓ_0 rule implies that we add two new vertices for each trivial path at a vertex labeled 1. Since $t(5) = 14$, we then need to add ten additional vertices. Since there are two paths of length 1, we have $\ell_1 = 5$. These generalized action graphs are shown below in Figure 5.

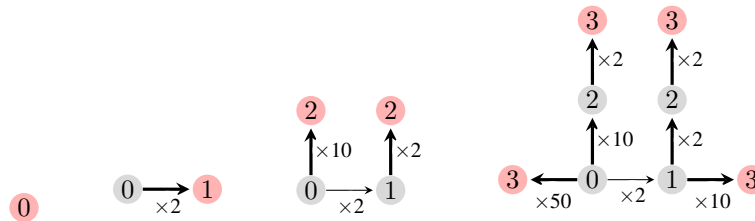


Figure 5: Attempted generalized action graphs for the internal triangle numbers

To create the next generalized action graph, note that we have fourteen vertices labeled 2 with trivial edges. By the path length rule ℓ_0 , we add two vertices labeled 3 for each vertex labeled 2, giving us 28 new vertices total. We have four paths of length one starting at vertices labeled 1, and ten paths of length

one starting at vertex 0. Since $\ell_1 = 5$, we add $4 \times 5 = 20$ new vertices with edges from the vertices labeled 1 and $10 \times 5 = 50$ new vertices with edges from vertex 0. Recall that $r(6) = 72$, but we already added ninety-eight ($28 + 20 + 50$) new vertices to this generalized action graph. Because of this contradiction, the internal triangle numbers cannot be modeled using generalized action graphs. \square

5 The super Catalan numbers

The super Catalan numbers are a generalized form of the Catalan numbers with two arguments, m and n .

Definition 5.1 ([1], A17). The **super Catalan numbers** are defined by

$$S(m, n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

When $m = 0$, the first few numbers in the sequence are 1, 2, 6, 20, and 70.

We wish to construct generalized action graphs for the sequence of super Catalan numbers where $m = 0$. In the rest of this section, we will state a method for constructing a sequence of directed graphs, conjecture that these graphs are indeed generalized action graphs for the sequence $S(0, n)$, and discuss our progress towards proving this conjecture.

Definition 5.2. We construct the sequence **generalized action graphs**, denoted $\{G_n\}$, for the super Catalan numbers as sequence of directed graphs defined inductively in the following way. The graph G_0 is a single vertex labeled 0. To construct G_{n+1} from G_n , consider each vertex v in G_n . For each $0 \leq \ell \leq n$, add $p(v, \ell) \cdot \frac{2}{2^\ell}$ new vertices labeled $n + 1$ with edges from v , where $p(v, \ell)$ is the number of paths of length ℓ from v to vertices labeled n in G_n .

For an example of the construction of G_0, G_1, G_2 , and G_3 see Figure 6. Note that we use the condensed notation introduced in Notation 5.

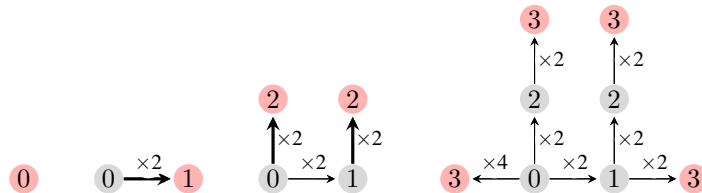


Figure 6: Generalized action graphs G_0 through G_3 for super Catalan numbers

Definition 5.2 satisfies Axiom 3, since each leaf in G_n will be labeled n and will have a path of length 0 to itself, and thus will have at least two new descendants, which will be labeled $n + 1$, in G_{n+1} .

Conjecture 5.2.1. Each graph G_n in the sequence of graphs constructed in Definition 5.2 satisfies Axiom 1 and Axiom 2.

In support of this conjecture, consider the following example. In Figure 7,

we use colors to show that each subtree of G_3 is isomorphic to a previous generalized action graph. Because the graphs are drawn in condensed notation, the multiplier on the edge from the 0 vertex indicates the number of subtrees that are isomorphic to a previous generalized action graphs. In particular, considering the subtrees rooted at vertices adjacent to the zero vertex: G_3 has two subtrees isomorphic to G_2 , two subtrees isomorphic to G_1 , and four subtrees isomorphic to G_0 , each with the labels of their vertices shifted up by an appropriate amount.

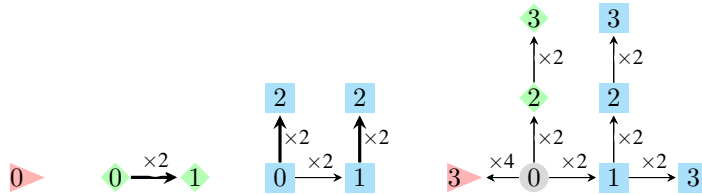


Figure 7: Highlighting the isomorphic subtrees of G_3

We will discuss our progress towards proving Conjecture 5.2.1. The data found in $\{G_n\}$ can be encoded into tables, which we call n -tables.

Let $K_{\ell,v,n}$ be the number of paths of length ℓ in G_n that start at a vertex labeled v and end at a vertex labeled n . For a given n , the table of $K_{\ell,v,n}$ for all values of ℓ and v is called the n -table.

Figure 8 shows G_3 , the conjectured generalized action graph for $S(0,3)$ in condensed notation, and Table 4 is the corresponding n -table. For example, to find $K_{1,2,3}$, count the paths of length 1 from vertices labeled 2 to vertices labeled 3 by taking the $2 \times 2 = 4$ paths from the leftmost 2 to vertices labeled 3, and adding the $2 \times 2 \times 2 = 8$ paths from the rightmost 2 to vertices labeled 3. Together that is 12, as seen in the corresponding entry of Table 4.

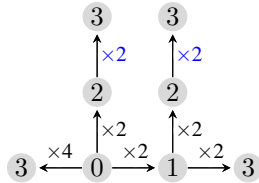


Figure 8: Conjectured G_3

$n = 3$	$v = 0$	$v = 1$	$v = 2$	$v = 3$
$\ell = 0$	0	0	0	20
$\ell = 1$	4	4	12	0
$\ell = 2$	8	8	0	0
$\ell = 3$	8	0	0	0

Table 4: $K_{\ell,v,3}$ values for G_3

Each entry in the n -table is completely determined by the graph G_n that it represents. Assuming Conjecture 5.2.2 is true, we found a way to compute the entries without having to count each path from each physically drawn generalized action graph:

- When $\ell = 0$, $K_{0,n,v} = 0$ for all v values except when $v = n$; in that case $K_{0,n,n} = S(0,n)$.
- For $\ell > 0$, entry $K_{\ell,v,n}$ in the n -table can be found using Conjecture 5.2.2, which uses the entries from the corresponding column of the $(n - 1)$ -table for the graph G_{n-1} .

Conjecture 5.2.2. The number of paths of length ℓ in G_{n+1} that start at a vertex labeled v and end at a vertex labeled $n + 1$ is given by

$$K_{\ell,v,n+1} = \sum_{i=0}^{n+1-\ell-v} \frac{2}{2^i} K_{\ell-1+i,v,n}.$$

We have checked that Conjecture 5.2.2 holds up through the 7-tables, but for larger n the generalized action graphs and the associated tables become cumbersome to both create and count. As an example of how to use Conjecture 5.2.2, consider the n -tables for the graphs of $S(0,3)$ and $S(0,4)$ in Tables 5 and 6 respectively.

$\ell \backslash v$	0	1	2	3
$\ell = 0$	0	0	0	20
$\ell = 1$	4	4	12	0
$\ell = 2$	8	8	0	0
$\ell = 3$	8	0	0	0

Table 5: $K_{\ell,v,3}$ values for G_3

$\ell \backslash v$	0	1	2	3	4
$\ell = 0$	0	0	0	0	70
$\ell = 1$	10	8	12	40	0
$\ell = 2$	20	16	24	0	0
$\ell = 3$	24	16	0	0	0
$\ell = 4$	16	0	0	0	0

Table 6: $K_{\ell,v,4}$ values for G_4

We will show how to compute $K_{1,0,4} = 10$ from Table 6 using Conjecture 5.2.2:

$$\begin{aligned}
 K_{1,0,4} &= \sum_{i=0}^3 \frac{2}{2^i} K_{i,0,3} \\
 &= \frac{2}{2^0}(0) + \frac{2}{2^1}(4) + \frac{2}{2^2}(8) + \frac{2}{2^3}(8) \\
 &= 10.
 \end{aligned}$$

If Conjecture 5.2.2 holds, we believe that Conjecture 5 can be used to compute the next super Catalan number, $S(0, n + 1)$, using the $K_{\ell,v,n}$ values from the n -table for the graph of $S(0, n)$, which would show the graph G_n satisfies Axiom 1.

The subsequent super Catalan number can be computed from the n -table of its previous action graph via

$$S(0, n + 1) = \sum_{\ell=0}^n \left(\frac{2}{2^\ell} \sum_{v=0}^n K_{\ell,v,n} \right).$$

This conjecture uses the sums of each column from Table 5. We will check that Conjecture 5 is true for $n = 3$:

$$\begin{aligned}
 S(0, 4) &= \sum_{\ell=0}^3 \left(\frac{2}{2^\ell} \sum_{v=0}^3 K_{\ell,v,3} \right) \\
 &= \frac{2}{2^0} \sum_{v=0}^3 K_{0,v,3} + \frac{2}{2^1} \sum_{v=0}^3 K_{1,v,3} + \frac{2}{2^2} \sum_{v=0}^3 K_{2,v,3} + \frac{2}{2^3} \sum_{v=0}^3 K_{3,v,3} \\
 &= 2(0 + 0 + 0 + 20) + 1(4 + 4 + 12 + 0) + \frac{1}{2}(8 + 8 + 0 + 0) \\
 &\quad + \frac{1}{4}(8 + 0 + 0 + 0) \\
 &= 70,
 \end{aligned}$$

which is known to be the super Catalan number $S(0, 4)$.

6 Acknowledgements

This project was supported by a grant from the Center for Undergraduate Research in Mathematics, which is funded by the National Science Foundation DMS awards 0636648, 1148695, and 1722563. Thanks also to Finn Richert and Maddie Dobrocky for their work in the early stages of these projects

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