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# Odd Graceful Labelings of Prism Graphs

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## Abstract

Odd graceful labelings of a graph are a variation of a graceful labeling. In each, the vertices are uniquely labeled with integers, and edges are assigned the difference between the incident vertex labels. For a graph with $m$ edges, the goal of a graceful labeling is to have distinct edge labels 1 to $m$, while an odd graceful labeling has odd edge labels from 1 to $2m-1$. In this paper we construct odd graceful labelings of prism graphs, denoted $C_n \times P_2$, when $n$ is even using the cases of $n = 6k, 6k+2$, and $6k+4$, which require similar but altered labelings.

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# 1 Introduction

A graceful labeling is a graph labeling that was developed over 50 years ago and which continues to be an active area of research. Graceful labelings were initially presented by Rosa [12], although he referred to them as a $\beta$*-labeling*. They were introduced to help decompose a complete graph into isomorphic subgraphs as an attempt to solve a problem known as Ringel's conjecture [11], which was made in 1963 and proven in 2021 [10]. Applications of this labeling, which was later called *graceful labeling* by Golomb [6], include other mathematical problems in coding theory and combinatorics, real-world problems involving seating arrangements, and science applications in x-ray crystallography and database management [1].

Graceful labelings are created by assigning distinct integers to vertices, with the absolute difference between these integers forming unique edge labels, while staying within a restricted set of possible labels. It is formally stated in Definition 1.1.

**Definition 1.** A *graceful labeling* of a graph with $m$ edges consists of the following:

1. Vertices are assigned distinct integers between 0 and $m$ using a function $\ell : V \to \{0, 1, \ldots, m\}$.
2. An edge $xy$ is assigned an integer by the function $\ell : E \to \{1, 2, \ldots, m\}$ with $\ell(xy) = |\ell(x) - \ell(y)|$, resulting in a set of edge labels equal to the set $\{1, 2, \ldots, m\}$.

If a graceful labeling exists for a particular graph $G$, then we say $G$ is *graceful*. Figure 1 shows that the cycle on four vertices is graceful.

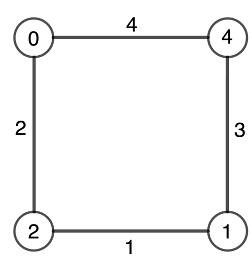


Figure 1: The cycle $C_4$ with a graceful labeling

Graceful labelings have been studied in numerous papers for various classes of graphs. For instance, cycles of length $n$, with $n \not\equiv 0$ or $3 \pmod 4$, were shown to be graceful in [12]. In addition, wheel graphs [2], pendant graphs [7], prism graphs [3], and certain sizes of generalized or stacked prism graphs (also called cylindrical grids) [8] are also graceful. Gallian's survey paper [4] provides an extensive list of other graceful graphs such as grids, helms, complete bipartite graphs, and many classes of trees. Note that it is still an open problem of whether all trees are graceful.

# 2 Odd Graceful Labelings

An odd graceful labeling is a variation of a graceful labeling where the first $m$ odd integers are required as the labels of the edges. This means there must be more integer options for the vertices in order to obtain these edge labels, causing the expression $2m-1$ to replace $m$ as the largest label on any vertex or edge. This labeling was introduced by Gnanajothi [5] and is formally stated as follows:

**Definition 2.** An *odd graceful labeling* of a graph with $m$ edges consists of the following:

1. Vertices are assigned distinct integers from 0 to $2m-1$ using a function $\ell : V \to \{0,1,\ldots,2m-1\}$.

2. An edge $xy$ is assigned an integer by the function $\ell : E \to \{1,3,\ldots,2m-1\}$ with $\ell(xy) = |\ell(x) - \ell(y)|$, resulting in a set of edge labels equal to the set $\{1,3,\ldots,2m-1\}$.

If a graph $G$ has an odd graceful labeling, then we refer to $G$ as being *odd graceful*. Figure 2 shows that the cycle on four vertices is odd graceful.

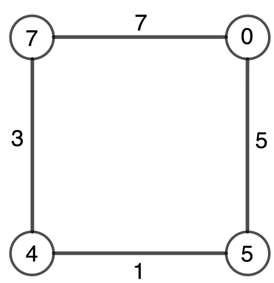


Figure 2: The cycle $C_4$ with an odd graceful labeling

Gnanajothi [5] proved a connection between odd graceful labelings and another graph labeling known as $\alpha$-labelings, particularly that every graph with an $\alpha$-labeling has an odd graceful labeling. An $\alpha$-labeling [12] is a graceful labeling $\ell$ with an additional condition that there is an integer $k$ such that given an edge $uv$, either $\ell(u) \leq k < \ell(v)$ or $\ell(v) \leq k < \ell(u)$. There are graphs, for example cycles of length $n \equiv 2 \pmod 4$, that are odd graceful, but for which an $\alpha$-labeling does not exist.

Odd graceful labelings have been constructed for many classes of graphs, starting in [5] with all paths $P_n$, cycles $C_n$ with even $n$, all complete bipartite graphs $K_{m,n}$, and any tree with up to 10 vertices. Additionally, all grid graphs were shown to have odd graceful labelings [13]. Gallian [4] provides further classes of graphs that have been shown to be odd graceful, and since graphs with $\alpha$-labelings are known to also have odd graceful labelings, the known results on that type of labeling add to the list of known odd graceful graphs.

A large set of graphs are quickly eliminated from consideration for odd graceful labelings, particularly those containing an odd cycle as a subgraph. Note that this is equivalent to requiring graphs to be bipartite in order for an odd graceful labeling to possibly exist. The following was originally proven by Gnanajothi [5] and is important in limiting the cases of prism graphs that we will investigate.

**Theorem 3** (Gnanajothi [5]). *A graph $G$ is not odd graceful if it contains an odd cycle as a subgraph.*

In order to create odd edge labels, the vertex labels must alternate between even and odd integers. Following this pattern in odd cycles will inevitably result in an even difference for the final edge no matter the choice for the last vertex label.

The cycle graph, denoted $C_n$, has $n$ vertices $v_1, v_2, \ldots, v_n$ as well as $n$ edges of the form $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $v_n v_1$. The following odd case is an immediate consequence of Theorem 3.

**Corollary 4.** *The cycle graph $C_n$ is not odd graceful if $n$ is odd.*

Gnanajothi [5] proved the following for the even case of the cycle graph. Although we do not show the full proof, we provide an odd graceful labeling for one case of an even cycle since it gives motivation for the approach we will take for prism graphs.

**Theorem 5** (Gnanajothi [5])**.** *The cycle graph $C_n$ is odd graceful if $n$ is even.*

The following labeling function for the vertices results in an odd graceful labeling for the case of $n = 4k+2$ for some integer $k \geq 1$:

$$\ell(v_i) = \begin{cases} i-2, & i = 2, 4, \ldots, 4k \\ i, & i = 4k+2 \\ 8k+4-i, & i = 1, 3, \ldots, 2k+1 \\ 8k+2-i, & i = 2k+3, 2k+5, \ldots, 4k+1. \end{cases}$$

An example for $n = 10$, where $k = 2$, can be seen in Figure 3. Notice how the labels on the odd-indexed vertices decrease by 2 each time until $i = 2k+3 = 7$, where it drops by 4. This skipping of a vertex label is essential to make sure the last edge $v_{10}v_1$ doesn't repeat a label. To label the different cases of prism graphs in the upcoming section, we make use of similar shifts in the vertex labels to skip edge labels that are assigned at the end of the cycles.

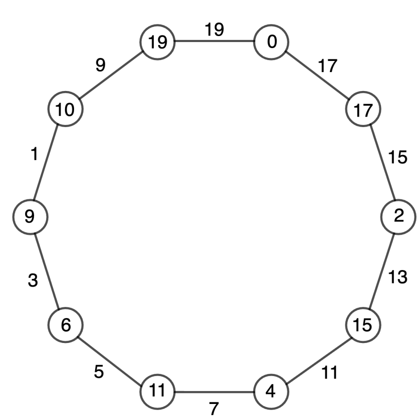


Figure 3: The cycle graph $C_{10}$ with an odd graceful labeling

## 3 Prism Graphs

Prism graphs were demonstrated in [3] to have an $\alpha$-labeling, and as previously discussed, all graphs with an $\alpha$-labeling are odd graceful. We aim in this section to explicitly construct an odd graceful labeling for each case of prism graphs when $n$ is even. These graphs consist of two cycles of length $n$ with additional edges connecting each vertex on the interior cycle to a corresponding one in the exterior cycle. In the example in Figure 4, each cycle has $n = 8$ vertices.

Let $C_n \times P_2$, for integers $n \geq 3$, denote the prism graph for a cycle of length $n$. The product notation represents the Cartesian product of two graphs, $G \times H$. Informally, this product creates a new graph with $|V(H)|$ copies of $G$, with the copies of each vertex of $G$ being connected in the manner of the edges in $H$. This means that for the prism graph, we get $|V(P_2)| = 2$ cycles $C_n$, in which we call the vertices $v_1, v_2, \ldots, v_n$ and $x_1, x_2, \ldots, x_n$. Edges are of the form $v_i v_{i+1}$ and $x_i x_{i+1}$ for integers $i$ with $1 \leq i \leq n-1$, along with $v_1 v_n$ and $x_1 x_n$ within those two copies of the cycle, and also $v_i x_i$ with $1 \leq i \leq n$ as a path $P_2$ is formed between the cycles. In upcoming figures, the vertices $v_1, v_2, \ldots, v_n$ will form the interior cycle and $x_1, x_2, \ldots, x_n$ the exterior cycle.

Prism graphs have also previously been shown in [2] to have a graceful labeling for all cycle lengths $n$. Furthermore, using longer paths than $P_2$ creates the stacked prism $C_n \times P_m$ with $m \geq 2$. The stacked prism $C_n \times P_m$ was shown in [2] to be graceful for all $n$ when $m$ is even and some small cases of $n$ when $m$ is odd.

As an example, an odd graceful labeling of $C_8 \times P_2$ is shown in Figure 4. Observe that this labeling fulfills Definition 2 as the vertices are labelled distinctly with integers between 0 and 47, and the edge labels are exactly $1, 3, \ldots, 47$. Every prism graph is graceful, but we show not every prism graph is odd graceful. The next four results combine to show $C_n \times P_2$ is odd graceful if and only if $n$ is even.

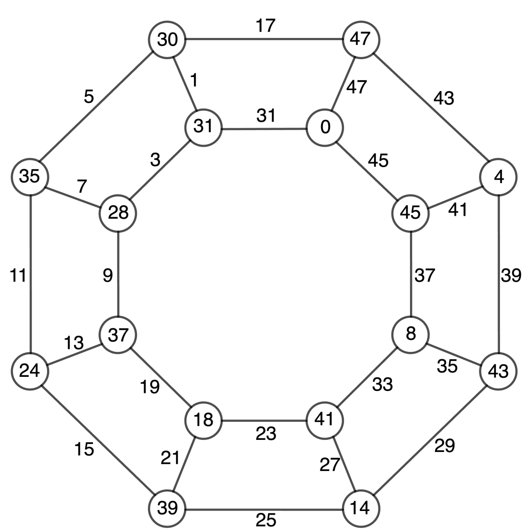


Figure 4: The prism graph $C_8 \times P_2$ with an odd graceful labeling

**Corollary 6.** *The prism graph $C_n \times P_2$ is not odd graceful for any odd integer $n \geq 3$.*

*Proof.* This is immediately true by Theorem 3 since the vertices $v_1, v_2, \ldots, v_n$ form an odd cycle. □

While the odd case of $n$ results in the prism graph not having an odd graceful labeling, we will show how to create one for any even $n$. Figures 4 and 5 show odd graceful labelings of two sizes of prisms. The labeling strategy is similar initially in which the edge labels decrease by 2 as you follow $v_1x_1$, $v_1v_2$, $x_1x_2$, $v_2x_2$, $x_2x_3$, $v_2v_3$, etc. If this pattern were to continue around the prism, we would repeat edge labels when $v_n$ and $x_n$ connect to $v_1$ and $x_1$. We therefore have to skip two labels at specific edges to successfully construct an odd graceful labeling. The difficulty that transpires is that the edges where the skipped labels are needed occur at different locations depending on whether the even $n$ is of the form $6k$, $6k+2$, or $6k+4$. Thus, different labeling functions are needed as we develop odd graceful labelings for each case of even-sided prism graphs. Figure 5 shows an example of the first case of an odd graceful labeling for $n = 12$, or $k = 2$.

**Theorem 7.** *The prism graph $C_{6k} \times P_2$ has an odd graceful labeling for all $k \geq 1$ using the following vertex labeling function:*

$$\ell(v_i) = \begin{cases} 36k+1-2i, & i = 2,4,\ldots,6k \\ 4i-4, & i = 1,3,\ldots,2k-1 \\ 4i-2, & i = 2k+1, 2k+3, \ldots, 4k-1 \\ 4i, & i = 4k+1, 4k+3, \ldots, 6k-1, \end{cases}$$

$$\ell(x_i) = \begin{cases} 36k+1-2i, & i = 1,3,\ldots,6k-1 \\ 4i-4, & i = 2,4,\ldots,2k \\ 4i-2, & i = 2k+2, 2k+4, \ldots, 4k \\ 4i, & i = 4k+2, 4k+4, \ldots, 6k. \end{cases}$$

*Proof.* First we observe that the number of edges in $C_{6k} \times P_2$ is $3(6k) = 18k$. Then by Definition 2, the largest edge and vertex label allowed is $2 \cdot 3n - 1 = 2(18k) - 1 = 36k - 1$. Therefore, we need to show this labeling has distinct vertex labels within the set $\{0, 1, \ldots, 36k-1\}$ with the edge labels $\ell(uv) = |\ell(u) - \ell(v)|$ being $1, 3, \ldots, 36k-1$.

Focusing first on the vertex labels, it is clear that our labels are between 0 and $36k-1$ since the largest label is $\ell(x_1) = 36k+1-2\cdot 1 = 36k-1$ and the smallest label is $\ell(v_1) = 4\cdot 1 - 4 = 0$. To show the vertex labels are distinct, we consider the sequences formed by the different lines of the two parts of our labeling function. The first line of $\ell(x_i)$ is the sequence of odd integers

$$36k-1, 36k-5, \ldots, 24k+3$$

since $36k+1-2(6k-1) = 24k+3$. Then the first line of $\ell(v_i)$ is the sequence

$$36k-3, 36k-7, \ldots, 24k+1.$$

We see that these two sequences consist of distinct odd integers.

The remaining labels will all be even integers, starting with the second line of $\ell(v_i)$:

$$0, 8, \ldots, 8k-8$$

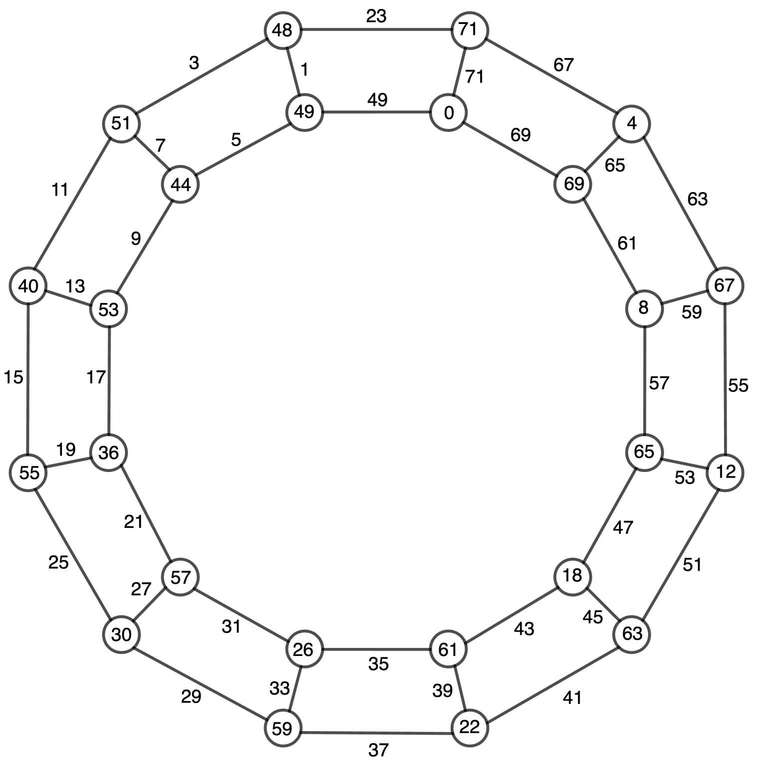


Figure 5: The prism graph $C_{12} \times P_2$ with an odd graceful labeling

since $4(2k-1)-4 = 8k-8$, while the second line of $\ell(x_i)$ is the following:

$$4, 12, \ldots, 8k-4.$$

Note that these two sequences have $k$ entries, so for the example in Figure 5 where $k = 2$, the sequences are merely $0, 8$ and $4, 12$. We list three terms of each labeling sequence to demonstrate the difference between terms, but keep in mind some condense down to only 1 or 2 terms when $k$ is small.

We continue by observing that the third line of $\ell(v_i)$ is the sequence

$$8k+2, 8k+10, \ldots, 16k-6$$

by simplifying $4(2k+1)-2, 4(2k+3)-2, \ldots, 4(4k-1)-2$. The third line of $\ell(x_i)$ is the sequence

$$8k+6, 8k+14, \ldots, 16k-2.$$

The fourth line of $\ell(v_i)$ is the sequence

$$16k+4, 16k+12, \ldots, 24k-4.$$

Finally, the fourth line of $\ell(x_i)$ is the sequence

$$16k+8, 16k+16, \ldots, 24k.$$

From those six lines of our function, one can see that they are nonoverlapping sequences of even labels. Therefore, all of our vertex labels are distinct integers in $\{0, 1, \ldots, 36k-1\}$, which satisfies the first part of the definition of an odd graceful labeling.

We now consider the set of labels on the edges, where we need to show this set is $\{1, 3, 5, \ldots, 36k-1\}$. We will examine sequences of labels for specific subsets of edges whose incident vertex pairs use the same lines of our vertex labeling function. First consider a set of edges connecting vertices between the two cycles: $v_1x_1, v_2x_2, \ldots, v_{2k}x_{2k}$, stopping just before the first shift in vertex labels between the second and third lines of the labeling function. For odd $i$, combining line 2 of $\ell(v_i)$ and line 1 of $\ell(x_i)$,

$$\begin{aligned}\ell(v_ix_i) &= |(4i-4)-(36k+1-2i)| \\ &= |6i-5-36k| \\ &= 36k-6i+5.\end{aligned}$$

For even $i$, notice that $\ell(v_i) = \ell(x_{i-1})$ and $\ell(x_i) = \ell(v_{i-1})$. Therefore, we get the same result for the edge label after applying the absolute value to the difference. In this particular case, $\ell(v_ix_i) = |(36k+1-2i)-(4i-4)| = 36k-6i+5$. The edge labels $\ell(v_ix_i)$ for $1 \le i \le 2k$ form the sequence

$$36k-1, 36k-7, \ldots, 24k+5. \tag{1}$$

Next, we consider the edges, $v_1v_2, x_2x_3, v_3v_4, \ldots, v_{2k-1}v_{2k}, x_{2k}x_{2k+1}$ that alternate between the two cycles. For odd $i$, we obtain the following from line 2 of $\ell(v_i)$ and line 1 of $\ell(v_i)$ with $i+1$ substituted for $i$:

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(4i-4)-[36k+1-2(i+1)]| \\ &= |6i-36k-3| \\ &= 36k-6i+3.\end{aligned}$$

Observe that for even $i$, $\ell(x_ix_{i+1})$ results in the same expression as above. For integers $i$ with $1 \le i \le 2k$, this develops the sequence of labels

$$36k-3, 36k-9, \ldots, 24k+3. \tag{2}$$

Then we consider the edges $x_1x_2, v_2v_3, x_3x_4, \ldots, v_{2k-2}v_{2k-1}, x_{2k-1}x_{2k}$ with the opposite alternating pattern. Using lines 1 and 2 from $\ell(x_i)$, we have for odd $i$

$$\begin{aligned}\ell(x_ix_{i+1}) &= |(36k+1-2i)-[4(i+1)-4]| \\ &= 36k-6i+1,\end{aligned}$$

which is identical for $\ell(v_iv_{i+1})$ when $i$ is even. This results in the following sequence as $i = 1$ to $2k-1$:

$$36k-5, 36k-11, \ldots, 24k+7. \tag{3}$$

Observe that the sequences in (1), (2), and (3) do not overlap and include every odd integer $24k+3$ to $36k-1$.

Now we consider three similar sets of edges before the last shift in vertex labels, starting with $v_{2k+1}x_{2k+1}, v_{2k+2}x_{2k+2}, \ldots, v_{4k}x_{4k}$. For odd $i$,

$$\begin{aligned}\ell(v_ix_i) &= |(36k+1-2i)-(4i-2)| \\ &= 36k-6i+3.\end{aligned}$$

The same expression is obtained in the even $i$ case because of the absolute value. This results, as $i = 2k+1$ to $4k$, in the sequence of labels

$$24k-3, 24k-9, \ldots, 12k+3. \tag{4}$$

Next, we consider the edges $v_{2k+1}v_{2k+2}, x_{2k+2}x_{2k+3}, v_{2k+3}v_{2k+4}, \ldots, v_{4k-1}v_{4k}, x_{4k}x_{4k+1}$, which have the labels

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(4i-2) - [36k+1-2(i+1)]| \\ &= 36k-6i+1,\end{aligned}$$

and likewise for $\ell(x_ix_{i+1})$. This develops the following sequence of labels as $i = 2k+1$ to $4k$:

$$24k-5, 24k-11, \ldots, 12k+1. \tag{5}$$

Next, the labels on edges $v_{2k}v_{2k+1}, x_{2k+1}x_{2k+2}, v_{2k+2}v_{2k+3}, \ldots, v_{4k-2}v_{4k-1}, x_{4k-1}x_{4k}$ have the formula

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(36k+1-2i) - [4(i+1)-2]| \\ &= 36k-6i-1,\end{aligned}$$

which matches each $\ell(x_ix_{i+1})$. As we substitute $i = 2k$ to $4k-1$, the sequence that develops is

$$24k-1, 24k-7, \ldots, 12k+5. \tag{6}$$

One can see that the sequences in (4), (5), and (6) include the distinct odd integers from $12k+1$ to $24k-1$. However, note the label $24k+1$ has been skipped thus far.

Now we consider the edges $v_{4k+1}x_{4k+1}, v_{4k+2}x_{4k+2}, \ldots, v_{6k}x_{6k}$. For odd $i$,

$$\begin{aligned}\ell(v_ix_i) &= |(36k+1-2i) - 4i| \\ &= 36k-6i+1,\end{aligned}$$

and we get the same formula for even $i$. This results in the following sequence of labels as $i = 4k+1$ to $6k$:

$$12k-5, 12k-11, \ldots, 1. \tag{7}$$

Next, we consider the edges $v_{4k}v_{4k+1}, x_{4k+1}x_{4k+2}, v_{4k+2}v_{4k+3}, \ldots, v_{6k-2}v_{6k-1}, x_{6k-1}x_{6k}$, which have the labels

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(36k+1-2i) - [4(i+1)]| \\ &= 36k-6i-3.\end{aligned}$$

The same expression results from calculating $\ell(x_ix_{i+1})$, which as $i = 4k$ to $6k-1$ develops the sequence of labels

$$12k-3, 12k-9, \ldots, 3. \tag{8}$$

Finally, the edges $v_{4k+1}v_{4k+2}, x_{4k+2}x_{4k+3}, v_{4k+3}v_{4k+4}, \ldots, x_{6k-2}x_{6k-1}, v_{6k-1}v_{6k}$ are assigned

$$\begin{aligned}\ell(v_iv_{i+1}) &= |4i - [36k+1-2(i+1)]| \\ &= 36k-6i-1.\end{aligned}$$

For $4k+1 \leq i \leq 6k-1$, the labeling sequence is

$$12k-7, 12k-13, \ldots, 5. \tag{9}$$

The sequences in (7), (8), and (9) include every odd integer 1 to $12k-3$. Note that a second label was skipped, particularly $12k-1$.

The final edge labels are

$$\ell(v_{6k}v_1) = |[36k+1-2(6k)] - (4\cdot 1 - 4)| = 24k+1$$

and

$$\ell(x_{6k}x_1) = |4(6k) - (36k + 1 - 2 \cdot 1)| = |24k - 36k + 1| = 12k - 1.$$

Since these are exactly the two labels that were skipped between the previous sequences, we have shown that the set of edge labels includes the entire sequence of odd integers from 1 to $36k-1$. Thus, we have proven that this is an odd graceful labeling when the length of the cycle is $6k$. □

We now proceed to the case of the cycle length $n$ being of the form $6k+2$, an example of which was in Figure 4 when $n = 8$.

**Theorem 8.** *The prism graph $C_{6k+2} \times P_2$ has an odd graceful labeling for all $k \geq 1$ using the following vertex labeling function:*

$$\ell(v_i) = \begin{cases} 36k + 13 - 2i, & i = 2, 4, \ldots, 6k \\ 4i - 1, & i = 6k + 2 \\ 4i - 4, & i = 1, 3, \ldots, 2k + 1 \\ 4i - 2, & i = 2k + 3, 2k + 5, \ldots, 4k + 1 \\ 4i, & i = 4k + 3, 4k + 5, \ldots, 6k + 1, \end{cases}$$

$$\ell(x_i) = \begin{cases} 36k + 13 - 2i, & i = 1, 3, \ldots, 6k + 1 \\ 4i - 4, & i = 2, 4, \ldots, 2k \\ 4i - 2, & i = 2k + 2, 2k + 4, \ldots, 4k \\ 4i, & i = 4k + 2, 4k + 4, \ldots, 6k \\ 4i - 2, & i = 6k + 2. \end{cases}$$

*Proof.* First note that with $3n = 3(6k + 2) = 18k + 6$ edges, our largest edge and vertex label is $2(18k + 6) - 1 = 36k + 11$. Therefore, we first show that our vertex labels are distinct integers in the set $\{0, 1, \ldots, 36k + 11\}$. It is clear that our labels are within this set since the largest label is $\ell(x_1) = 36k + 13 - 2 \cdot 1 = 36k + 11$ and the smallest label is $\ell(v_1) = 4 \cdot 1 - 4 = 0$.

As in the $n = 6k$ result, to show the vertex labels are distinct, we consider the sequences formed by the different lines of the two parts of our labeling function. Table 1 shows the sequences of vertex labels, with the top rows containing the odd labels and the bottom portion displaying the even labels. One can observe that all vertices are included and the labeling sequences are nonoverlapping. Therefore, all of our vertex labels are distinct integers in $\{0, 1, \ldots, 36k + 11\}$, which satisfies the first part of the definition of an odd graceful labeling.

We now consider the labels on the edges, where we need to show this set is exactly the distinct odd integers $1, 3, 5, \ldots, 36k + 11$. For brevity, we will omit the details of obtaining the labeling sequences, as the differences in vertex labels are calculated similarly to those in the $n = 6k$ result. Table 2 shows the edge and labeling sequences.

Observe that all odd integers between 7 and $36k + 11$ are included within the edge labels in Table 2 with the exception of $12k + 5$ and $24k + 7$. The remaining edge labels involve at least one of the separately labeled vertices $v_{6k+2}$ or $x_{6k+2}$: $\ell(x_{6k+1}x_{6k+2}) = 5, \ell(v_{6k+1}v_{6k+2}) = 3$, $\ell(v_{6k+2}x_{6k+2}) = 1, \ell(v_{6k+2}v_1) = 24k + 7$, and $\ell(x_{6k+2}x_1) = 12k + 5$.

| Vertex sequence | Labeling sequence |
|---|---|
| $x_1, x_3, \ldots, x_{6k+1}$ | $36k+11, 36k+7, \ldots, 24k+11$ |
| $v_2, v_4, \ldots, v_{6k}$ | $36k+9, 36k+5, \ldots, 24k+13$ |
| $v_{6k+2}$ | $24k+7$ |
| $v_1, v_3, \ldots, v_{2k+1}$ | $0, 8, \ldots, 8k$ |
| $x_2, x_4, \ldots, x_{2k}$ | $4, 12, \ldots, 8k-4$ |
| $x_{2k+2}, x_{2k+4}, \ldots, x_{4k}$ | $8k+6, 8k+14, \ldots, 16k-2$ |
| $v_{2k+3}, v_{2k+5}, \ldots, v_{4k+1}$ | $8k+10, 8k+18, \ldots, 16k+2$ |
| $x_{4k+2}, x_{4k+4}, \ldots, x_{6k}$ | $16k+8, 16k+16, \ldots, 24k$ |
| $v_{4k+3}, v_{4k+5}, \ldots, v_{6k+1}$ | $16k+12, 16k+20, \ldots, 24k+4$ |
| $x_{6k+2}$ | $24k+6$ |

**Table 1:** The vertex labels for the prism graph $C_{6k+2} \times P_2$

These final five edge labels complete the entire sequence of odd integers from 1 to $36k+11$, include filling in the two skipped labels of $12k+5$ and $24k+7$. Thus, we have shown this is odd graceful labeling for when the cycle has length $6k+2$.

□

We conclude our examination of prism graphs with cycle lengths in which $n = 6k+4$. Figure 6 shows an example from this case of an odd graceful labeling of $C_{16} \times P_2$.

**Theorem 9.** *The prism graph $C_{6k+4} \times P_2$ has an odd graceful labeling for all $k \geq 0$ using the following vertex labeling function:*

$$\ell(v_i) = \begin{cases} 36k+25-2i, & i = 2, 4, \ldots, 2k \\ 36k+23-2i, & i = 2k+2, 2k+4, \ldots, 6k+2 \\ 4i-3, & i = 6k+4 \\ 4i-4, & i = 1, 3, \ldots, 4k+1 \\ 4i-2, & i = 4k+3, 4k+5, \ldots, 6k+3, \end{cases}$$

$$\ell(x_i) = \begin{cases} 36k+25-2i, & i = 1, 3, \ldots, 2k+1 \\ 36k+23-2i, & i = 2k+3, 2k+5, \ldots, 6k+3 \\ 4i-4, & i = 2, 4, \ldots, 2k \\ 4i-6, & i = 2k+2 \\ 4i-4, & i = 2k+4, 2k+6, \ldots, 4k+2 \\ 4i-2, & i = 4k+4, 4k+6, \ldots, 6k+2 \\ 4i-4, & i = 6k+4. \end{cases}$$

*Proof.* First, note that we have the situation of $k = 0$ included whereas previous results began with $k = 1$. This will result in some rows of the upcoming table being empty sequences; however, the labeling will be odd graceful in that single case as well.

| Edge sequence | Labeling sequence |
|---|---|
| $v_1x_1, v_2x_2, \ldots, v_{2k+1}x_{2k+1}$ | $36k+11, 36k+5, \ldots, 24k+11$ |
| $v_1v_2, x_2x_3, v_3v_4, \ldots, x_{2k}x_{2k+1}, v_{2k+1}v_{2k+2}$ | $36k+9, 36k+3, \ldots, 24k+9$ |
| $x_1x_2, v_2v_3, x_3x_4, \ldots, x_{2k-1}x_{2k}, v_{2k}v_{2k+1}$ | $36k+7, 36k+1, \ldots, 24k+13$ |
| $v_{2k+2}x_{2k+2}, v_{2k+3}x_{2k+3}, \ldots, v_{4k+1}x_{4k+1}$ | $24k+3, 24k-3, \ldots, 12k+9$ |
| $x_{2k+1}x_{2k+2}, v_{2k+2}v_{2k+3}, x_{2k+3}x_{2k+4}, \ldots, x_{4k-1}x_{4k}, v_{4k}v_{4k+1}$ | $24k+5, 24k-1, \ldots, 12k+11$ |
| $x_{2k+2}x_{2k+3}, v_{2k+3}v_{2k+4}, x_{2k+4}x_{2k+5}, \ldots, x_{4k}x_{4k+1}, v_{4k+1}v_{4k+2}$ | $24k+1, 24k-5, \ldots, 12k+7$ |
| $v_{4k+2}x_{4k+2}, v_{4k+3}x_{4k+3}, \ldots, v_{6k+1}x_{6k+1}$ | $12k+1, 12k-5, \ldots, 7$ |
| $x_{4k+1}x_{4k+2}, v_{4k+2}v_{4k+3}, \ldots, v_{6k}v_{6k+1}$ | $12k+3, 12k-3, \ldots, 9$ |
| $x_{4k+2}x_{4k+3}, v_{4k+3}v_{4k+4}, x_{4k+3}x_{4k+4}, \ldots, v_{6k-1}v_{6k}, x_{6k}x_{6k+1}$ | $12k-1, 12k-7, \ldots, 11$ |

**Table 2:** The edge labels for the prism graph $C_{6k+2} \times P_2$

With $3n = 3(6k+4) = 18k+12$ edges, the largest edge and vertex label would be $2(18k+12)-1 = 36k+23$. Then we first show that our vertex labels are distinct integers in $\{0, 1, \ldots, 36k+23\}$. The largest label is $\ell(x_1) = 36k+25-2\cdot 1 = 36k+23$ and the smallest label is $\ell(v_1) = 4\cdot 1 - 4 = 0$, so the vertex labels are all within the set.

Following the a similar process as the previous two cases, Table 3 shows the sequences of odd and even vertex labels. One can observe that all of our vertex labels are distinct integers in $\{0, 1, \ldots, 36k+23\}$, satisfying the first part of the definition of an odd graceful labeling.

| Vertex Sequence | Labeling Sequence |
|---|---|
| $x_1, x_3, \ldots, x_{2k+1}$ | $36k+23, 36k+19, \ldots, 32k+23$ |
| $x_{2k+3}, x_{2k+5}, \ldots, x_{6k+3}$ | $32k+17, 32k+13, \ldots, 24k+17$ |
| $v_2, v_4, \ldots, v_{2k}$ | $36k+21, 36k+17, \ldots, 32k+25$ |
| $v_{2k+2}, v_{2k+4}, \ldots, v_{6k+2}$ | $32k+19, 32k+15, \ldots, 24k+19$ |
| $v_{6k+4}$ | $24k+13$ |
| $v_1, v_3, \ldots, v_{4k+1}$ | $0, 8, \ldots, 16k$ |
| $x_2, x_4, \ldots, x_{2k}$ | $4, 12, \ldots, 8k-4$ |
| $x_{2k+4}, x_{2k+6}, \ldots, x_{4k+2}$ | $8k+12, 8k+20, \ldots, 16k+4$ |
| $v_{4k+3}, v_{4k+5}, \ldots, v_{6k+3}$ | $16k+10, 16k+18, \ldots, 24k+10$ |
| $x_{4k+4}, x_{4k+6}, \ldots, x_{6k+2}$ | $16k+14, 16k+22, \ldots, 24k+6$ |
| $x_{2k+2}$ | $8k+2$ |
| $x_{6k+4}$ | $24k+12$ |

**Table 3:** The vertex labels for the prism graph $C_{6k+4} \times P_2$

We now consider the labels on the edges. We need to show the set of edge labels is $\{1, 3, 5, \ldots, 36k+23\}$. Table 4 shows the edge and labeling sequences, which are

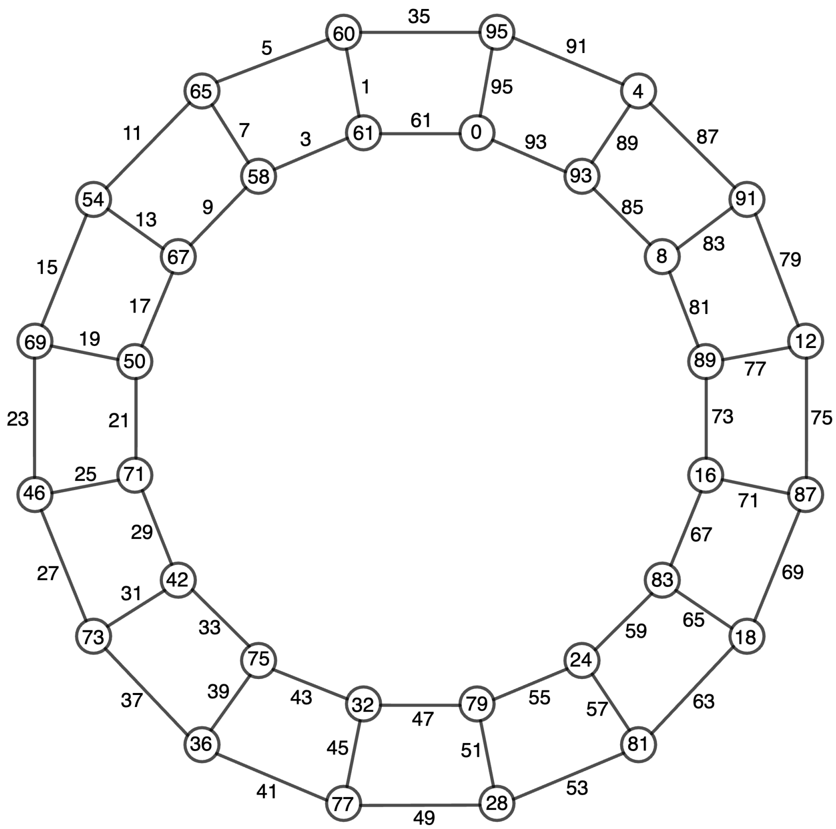

Figure 6: The prism graph $C_{16} \times P_2$ with an odd graceful labeling

calculated similarly to cases $n = 6k$ and $n = 6k+2$. This case does differ, however, in that we next have to consider some individual edges as shifts occur within our labeling function:

$$\ell(x_{2k+1}x_{2k+2}) = |36k+25-2(2k+1)-[4(2k+2)-6]| = 24k+21,$$
$$\ell(v_{2k+1}v_{2k+2}) = |[4(2k+1)-4)]-[36k+23-2(2k+2)]| = 24k+19,$$
$$\ell(v_{2k+2}x_{2k+2}) = |[36k+23-2(2k+2)]-[4(2k+2)-6]| = 24k+17,$$
$$\ell(x_{2k+2}x_{2k+3}) = |[4(2k+2)-6)]-[36k+23-2(2k+3)]| = 24k+15.$$

At this point, observe that Table 4 and the previous four labels include every odd integer 7 to $36k+23$, with the exception of $24k+13$ and $12k+11$ being skipped. There are five edges remaining that are incident on at least one of $v_{6k+4}$ or $x_{6k+4}$, which are labeled by $24k+13$ and $24k+12$, respectively. They are the following: $\ell(x_{6k+3}x_{6k+4}) = 5$, $\ell(v_{6k+3}v_{6k+4}) = 3$, $\ell(v_{6k+4}x_{6k+4}) = 1$, $\ell(v_{6k+4}v_1) = 24k+13$, and $\ell(x_{6k+4}x_1) = 12k+11$.

We have shown that the set of edge labels includes the entire sequence of odd integers from 1 to $36k+23$. Thus, we have shown that this is an odd graceful labeling when the cycle length is of the form $6k+4$. □

| Edge sequence | Labeling sequence |
|---|---|
| $v_1x_1, v_2x_2, \dots, v_{2k+1}x_{2k+1}$ | $36k+23, 36k+17, \dots, 24k+23$ |
| $x_1x_2, v_2v_3, x_3x_4, \dots, x_{2k-1}x_{2k}, v_{2k}v_{2k+1}$ | $36k+19, 36k+13, \dots, 24k+25$ |
| $v_1v_2, x_2x_3, v_3v_4, \dots, v_{2k-1}v_{2k}, x_{2k}x_{2k+1}$ | $36k+21, 36k+15, \dots, 24k+27$ |
| $v_{2k+3}x_{2k+3}, v_{2k+4}x_{2k+4}, \dots, v_{4k+2}x_{4k+2}$ | $24k+9, 24k+3, \dots, 12k+15$ |
| $v_{2k+2}v_{2k+3}, x_{2k+3}x_{2k+4}, v_{2k+4}v_{2k+5}, \dots, v_{4k}v_{4k+1}, x_{4k+1}x_{4k+2}$ | $24k+11, 24k+5, \dots, 12k+17$ |
| $v_{2k+3}v_{2k+4}, x_{2k+4}x_{2k+5}, v_{2k+5}v_{2k+6}, \dots, v_{4k+1}v_{4k+2}, x_{4k+2}x_{4k+3}$ | $24k+7, 24k+1, \dots, 12k+13$ |
| $v_{4k+3}x_{4k+3}, v_{4k+4}x_{4k+4}, \dots, v_{6k+3}x_{6k+3}$ | $12k+7, 12k+1, \dots, 7$ |
| $v_{4k+3}v_{4k+4}, x_{4k+4}x_{4k+5}, v_{4k+5}v_{4k+6}, \dots, v_{6k+1}v_{6k+2}, x_{6k+2}x_{6k+3}$ | $12k+5, 12k-1, \dots, 11$ |
| $v_{4k+2}v_{4k+3}, x_{4k+3}x_{4k+4}, v_{4k+4}v_{4k+5}, \dots, x_{6k+1}x_{6k+2}, v_{6k+2}v_{6k+3}$ | $12k+9, 12k+3, \dots, 9$ |

**Table 4:** The edge labels for the prism graph $C_{6k+4} \times P_2$

## 4 Open Problems

A natural extension of cycle graphs (which could be viewed as $C_n \times P_1$) and prism graphs ($C_n \times P_2$) would be to increase the value $m$ on $C_n \times P_m$, referred to as the stacked prism or cylindrical grid graph. Observe that as long as $n$ is even, all cycles within the graph would be of even length. Note that although various cases of $C_n \times P_m$ have been shown to be graceful (see Table 1 in [4]), it appears to remain an open problem for whether that is true for all $m$ and $n$. From an $\alpha$-labeling perspective, labelings were developed in [9] for all cases of even $n$ except for $C_{4k+2} \times P_{2\ell+1}$, which remains open.

Another way to view prism graphs is as the most basic version of a generalized Petersen graph, $GP(n,k)$ with $k < n/2$. Like prism graphs, $GP(n,k)$ has vertices $v_1, v_2, \dots, v_n$ and $x_1, x_2, \dots, x_n$. It also has edges of the form $v_ix_i$ from 1 to $n$ and $x_ix_{i+1}$ for $1 \le i \le n-1$ along with $x_nx_1$. Where it differs are the edges between the interior $v_i$ vertices. The prism graph, which is the case $GP(n,1)$, has edges $v_iv_{i+1}$, but the more general $GP(n,k)$ has edges $v_iv_{i+k}$. Similar to stacked prisms, progress has been made demonstrating $GP(n,k)$ is graceful for small $n$ and any $k < n/2$, but the question for larger $n$ and $k$ remains unresolved for both graceful and odd graceful labelings.

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