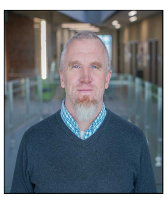
Mathematics Exchange, 19(1), Fall 2025, 31–45
DOI: 10.33043/7dkds89xqme
https://openjournals.bsu.edu/mathexchange

Generalized Bondage Number: The k-synchronous bondage number of a graph

Rey Anaya, Alvaro Belmonte, Nathan Shank, Elise Sinani, Bryan Walker*

Rey Anaya was raised in Bethlehem, Pennsylvania, and is a graduate of Freedom High School. In 2021, he earned a Bachelor of Science in Mathematics with a minor in Computer Science from Moravian College. Following graduation, he enlisted in the United States Air Force and is currently stationed at Whiteman Air Force Base, home of the B-2 Spirit Stealth Bomber.

Alvaro Belmonte earned an Associate's degree in Engineering from Northampton Community College in 2018, followed by a Bachelor of Science in Mathematics from Moravian College in 2020. He is now pursuing a Ph.D. in Mathematics at Johns Hopkins University, where his research focuses on category theory.



Nathan Shank is a Professor of Mathematics and the Louise E. Juley Chair in Sciences at Moravian University. He has been involved in REU projects for over a decade and enjoys exploring problems at the intersection of mathematics and computer science.

Elise Sinani graduated from Virginia Tech in 2020 with a Bachelor of Science degree in Mathematics. This research was completed as part of a Research Experience for Undergraduates (REU) during the summer before her senior year. Since graduation, she has focused on raising a family, including teaching mathematics to her children.

***Corresponding author:** shankn@moravian.edu

Bryan Walker received his Bachelor's degree in Mathematics and Physics in 2020 from Sewanee: The University of the South. He is currently a Ph.D. student in geometric analysis at the University of Tennessee, Knoxville. His primary research interests include mathematical physics and relativity.

Abstract

We investigate a generalization of the bondage number of a graph called the k-*synchronous bondage number*. The k-synchronous bondage number of a graph is the smallest number of edges that, when removed, increases the domination number by k. In this paper, we discuss the 2-synchronous bondage number and then generalize to the k-synchronous bondage number. We present k-synchronous bondage numbers for several graph classes and give bounds for general graphs. We propose this characteristic as a metric of the connectivity of a simple graph with possible uses in the field of network design and optimization.

1 Introduction

Graphs serve as a mathematical tool for analyzing networks where the vertices of graphs can represent stations, transmitters, people, computers, cell phones, or cities while the edges demonstrate the connections between these objects, such as railroads, power lines, friendships, computer connections, signals, or roads. Representing such networks as graphs allows us to apply the tools and properties of graph theory to real world problems thereby providing rigorously proven solutions.

An unreliable network is one that can be easily disrupted either maliciously or accidentally. To ensure the reliability of a network, an understanding of its purpose and sensitivities must be taken into account throughout its construction and preservation. Thus, we must determine the minimum requirements in order for a network to remain operational as well as which parts of the network can break down and thus cause a failure state.

When modeling networks with graphs, we can analyze different failure states and, therefore, quantify network reliability. For instance, if the network is operational as long as the graph is connected, then the graph is in a failure state once there are two or more components. Then we can evaluate the network's strength in comparison to other potential network designs by counting the minimum number of edges which must be removed in order to render the network inoperable. This measure of connectivity, known as the *edge connectivity* γ was first introduced by Beineke and Harary [3] as the minimum number of edges whose removal results in a disconnected graph. Sampathkum [10] extended this idea of connectivity to *g-component edge connectivity* which is the minimum number of edges that must be removed in order to create at least g components.

Connectivity and number of components are not the only properties that may be considered. As an example, the *component order edge connectivity* also extends

Beineke and Harary's idea by considering the removal of edges until every component of the graph has order less than a given positive integer k (see [4], [6], and [7]).

In this paper we will focus on edge removal and the impact on domination number of a graph. The *domination number* of a graph is the cardinality of the smallest set of vertices, V', so that every vertex not in V' is adjacent to at least one vertex in V'. The minimum number of edge removals that increases the domination number by one was first considered in [2] and called *dominating line-stability*. Later, in [5], the authors define this same concept to be the *bondage number* of a graph. Here we expand upon the concept of bondage number by investigating edge removals that increase the domination number by a specified number, k. We consider the graph, and hence the network it represents, to be in a failure state when the domination number increases by k. In order to achieve a failure state we are looking for an existential quantity rather than a universal one. That is to say, we are seeking the smallest positive integer m such that there exists an edge set of cardinality m which, when removed, creates a failure state-as opposed to finding the smallest positive integer M such that all edge sets of cardinality M, when removed, will create a failure state. Hence, for our purposes, if we remove any edge set with fewer than m edges, the network will be operational.

2 Background and Definitions

For our purposes, all graphs are assumed to be undirected simple graphs; that is, graphs with no loops and in which no two vertices share more than a single edge between them. If $G = (V,E)$ and $S \subseteq V$, then the *induced subgraph on S*, denoted $G[S]$, is the subgraph of G with vertex set S which includes all the edges in G whose vertices are in S. If $H = (V',E')$ we will denote the *disjoint union* of G and H as $G \oplus H = (V \cup V', E \cup E')$. For other graph theory notation and terminology we will use [12].

Given a graph $G = (V,E)$, a set of vertices $D \subseteq V$ is a *dominating set* if all vertices in V are either in D or adjacent to a vertex in D. The *domination number* of the graph, $\gamma(G)$, is the minimum size of a dominating set in G. The study of this characteristic is summarized efficiently in [8].

In 1990, Fink et al.[5] introduced, in its modern form, the idea of *bondage number*. The bondage number of a graph, denoted $b(G)$, is the size of the smallest subset of edges of G, which, when removed, increases the domination number. In 2013, Xu [13] defines a *bondage set* as any $\mathscr{E} \subseteq E$ such that $\gamma(G-\mathscr{E}) > \gamma(G)$. Furthermore a *minimum bondage set* is a bondage set with the smallest possible cardinality. We also know if $\mathscr{E}$ is a minimum bondage set, then $\gamma(G-\mathscr{E}) = \gamma(G)+1$ because the removal of any single edge can increase the domination number by at most one, and therefore, we can remove single edges until γ increases by exactly 1.

Given a set of graphs $\mathscr{G}$, we define the *minimum bondage number of* $\mathscr{G}$ as $b(\mathscr{G}) = \min\{b(G) : G \in \mathscr{G}\}$.

Given a graph $G = (V,E)$, we define the *bondage graphs of* G, denoted $\mathrm{BG}(G)$, to be the set of all graphs $G' = G-\mathscr{E}$ for some bondage set $\mathscr{E} \subseteq E$.

Similarly, given a graph $G = (V,E)$, we define the *minimum bondage graphs of* G, denoted $\mathrm{MBG}(G)$, to be the set of all graphs $G' = G-\mathscr{E}$ for any minimum bondage set $\mathscr{E} \subseteq E$.

We will say that G' is the result of a *bondage move* if $G' \in \mathrm{BG}(G)$. So, a bondage move is the removal of a bondage set from G and we will say the *size of the bondage move* is the size of the bondage set. Similarly, we will say that G' is the result of a *minimum bondage move* if $G' \in \mathrm{MBG}(G)$ and we will say the *size of a minimum bondage move* is the size of a minimum bondage set. A minimum bondage move is, therefore, the removal of a minimum bondage set from G.

So, if we are completing iterative bondage moves on some graph G, then $b(\mathrm{MBG}(G))$ is the minimum bondage number over the set of graphs resulting from all possible first bondage moves on G.

In this paper, we define *k-synchronous bondage set*, *minimum k-synchronous bondage set*, and *k-synchronous bondage number* which generalize a bondage set, minimum bondage set, and bondage number, respectively, by increasing the domination number by k. For k-synchronous bondage, we are removing a subset of edges simultaneously. Note that we can increase the domination number by making k minimum bondage moves which will result in a domination number of $\gamma(G)+k$, however, this will not always correspond to a minimum k-synchronous bondage set. Throughout this paper, we will assume that $|V| \geq k+\gamma(G)$ since the domination number cannot be larger than the order of the graph.

Definition 1 (k-synchronous bondage set)**.** Given a graph $G=(V,E)$ and a positive integer k, a set $\mathscr{E} \subseteq E$ is a k-synchronous bondage set if $\gamma(G-\mathscr{E}) = \gamma(G)+k$.

So a set of edges is a k-synchronous bondage set if the removal of the edges increases the domination number by k.

Definition 2 (minimum k-synchronous bondage set)**.** Given a graph $G=(V,E)$, a set $\mathscr{E} \subseteq E$ is a minimum k-synchronous bondage set if $\gamma(G-\mathscr{E}) = \gamma(G)+k$ and for all $E' \subseteq E$ with $|E'| < |\mathscr{E}|$, $\gamma(G-E') < \gamma(G)+k$.

Therefore, a minimum k-synchronous bondage set is a subset of the edge set whose removal increases the domination number by k and there does not exist a smaller subset that results in the domination number increasing by k.

Definition 3 (k-synchronous bondage number)**.** Given a graph $G=(V,E)$ and a positive integer k, the k-synchronous bondage number of G, denoted $Sb_k(G)$, is the size of a minimum k-synchronous bondage set.

Thus, the k-synchronous bondage number of G is the minimum number of edges that can be removed so that the domination number increases by k.

Connecting the previous definitions with bondage number, we see that the study of Sb_1 would be the same as the study of the bondage number, as indicated in the following proposition.

Proposition 4. *For any graph G, $b(G) = Sb_1(G)$.*

However, the k-synchronous bondage number differs from previous studies on $b(G)$ whenever $k \geq 1$. Our approach is novel in that we can specify any desired increase by setting the value of k. The combined size of two successive minimum bondage moves serves only as an upper bound for Sb_2, so it is essential to discuss several concepts regarding the bondage number for $k \geq 2$.

Note that for any graph G and edge e, $\gamma(G) \leq \gamma(G-e) \leq \gamma(G)+1$. This immediately implies the following proposition.

Proposition 5. *For any graph G and any positive integer k,*

$$Sb_k(G) \geq Sb_{k-1}(G) + 1 \geq k.$$

In this paper, we consider 2-synchronous bondage in section 3 where we analyze the relationship between 2-synchronous bondage and consecutive bondage moves. We also provide some bounds specific to 2-synchronous bondage for general graphs. In section 4, we proceed to demonstrate several properties of k-synchronous bondage numbers and provide proofs for the k-synchronous bondage of paths, cycles, trees and complete graphs. We conclude by presenting additional areas of potential research into the value of k-synchronous bondage numbers.

3 Properties of Sb_2

The value of studying Sb_k is two-fold. First, suppose we obtain $Sb_k(G)$ for some graph $G = (V,E)$ by iterative minimum bondage moves. This method will require us to analyze graphs outside of the families typically considered when studying $b(G)$. Beyond this, however, we find that for some graphs, G, $Sb_k(G)$ is less than the summation of the size of k iterative minimum bondage moves. For example, consider graph H in Figure 1 which consists of a complete graph on a set $A = \{a,b,c,d\}$, a complete graph on a set $B = \{e,f,g,h\}$ with the edge eh removed, a complete graph on a set $C = \{i,j,k,l\}$ with the edges ik and jl removed, along with the edges $E' = \{de,hi\}$, and f is a vertex of a complete graph on the set $F = \{f,f_2,f_3,\dots,f_n\}$ with $n \geq 5$, and g is a vertex of a complete graph on $G = \{g,g_2,g_3,\dots,g_m\}$ with $m \geq 5$. The induced subgraphs on F and G are complete subgraphs denoted by the dashed circles in Figure 1.

Notice $\gamma(H) = 5$ and a minimum dominating set is $D = \{d,f,g,i,k\}$. Now, $Sb_2(H) \leq 4$ by removing the 4 edges in $H[C]$; this removal results in vertices j,k, and l being included in every minimum dominating set as they are isolated vertices and the remaining connected component of H has a domination number of 4 and a minimum dominating set of $D' = \{d,f,g,i\}$.

It is straightforward to check that the removal of a single edge in H will not change the domination number, i.e. $b(H) \geq 2$. Therefore, by Proposition 5 we know $Sb_2(H) \geq 3$.

If $Sb_2(H) = 3$ and $b(H) \geq 2$ then there must be an edge set, E', of cardinality two and an edge ε so that $\gamma(H) = \gamma(H - E') - 1$ and $\gamma(H) = \gamma(H - E' - \{\varepsilon\}) - 2$.

It is easy to verify that the only way to increase the domination number with the removal of two edges is to remove two vertex-disjoint edges in $H[A]$. We give a brief justification below.

Consider $H[A]$, $H[C]$, and $H[B \cup F \cup G \cup \{d,i\}]$ which are three edge-disjoint subgraphs. The removal of any two edges in $H[C]$ will not increase the domination number. Also, we can see that the removal of any edge in $H[C]$ along with any other edge in H which is not in $H[C]$ will be dominated by $\{x,f',g',i,k\}$ for some $x \in A$, $f' \in F$, and $g' \in G$. The removal of any two edge in $H[F \cup G]$ will be dominated by $\{d,f',g',i,k\}$ for some vertex $f' \in F$ and $g' \in G$ because $n,m \geq 5$. The removal of any edge in $H[F]$ and any other edge in H which is not in $H[G]$ will be dominated by $\{x,f',g,i,k\}$ for some vertex $x \in A$ and $f' \in F$. Similarly, the removal of any edge in $H[G]$ and any other edge in H which is not in $H[F]$ will be dominated by $\{x,f,g',i,k\}$

for some vertex $x \in A$ and $g' \in G$. The removal of any two edges in $H[B \cup \{d,i\}]$ will be dominated by $D = \{d, f, g, i, k\}$ because e and h are adjacent to three vertices in D. And finally, the removal of an edge in $H[B \cup \{d,i\}]$ and an edge in $H[A]$ will be dominated by $\{x, f, g, i, k\}$ for some vertex $x \in A$.

Therefore, the only way to increase the domination number of H with the removal of two edges is to remove two edges in $H[A]$. For this to happen, the two edges in $H[A]$ must be vertex disjoint. Without loss of generality, we will assume we will remove $E' = \{ac, bd\}$.

Recall that the domination number can not increase by more than 1 with one edge removal by Proposition 5. Therefore, if $Sb_2(H) = 3$ there must be an edge ε in H so that $\gamma(H) = \gamma(H - \{ac, bd, \varepsilon\}) - 2$. However, a similar argument as posed for H can verify that $b(H - \{ac, bd\}) = 3$. Therefore $Sb_2(H) > 3$ and we can conclude $Sb_2(H) = 4$.

A single edge removal does not change the domination number of H and removing edges $E' = \{ac, bd\}$ increases the domination number by 1. Therefore we know $b(H) = 2$. However, $b(H - \{ac, bd\}) = 3$ which can be accomplished by the removal of $E'' = \{ad, ab, bc\}$. Thus, two successive minimum bondage moves requires the removal of 5 edges whereas $Sb_2(H) = 4$.

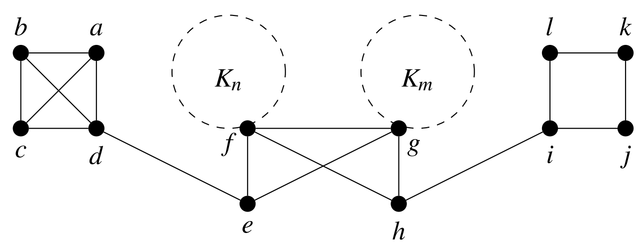


Figure 1: Graph H with $n, m \geq 4$.

Alternatively, we can show that there are circumstances under which the sum of the sizes of two successive minimum bondage moves is equal to Sb_2.

Theorem 6. *Let G be a graph. If* $b(MBG(G)) \leq 2$ *then* $Sb_2(G) = b(G) + b(MBG(G))$.

Proof. Since the size of two successive minimum bondage moves serves as an upper bound for $Sb_2(G)$, we will assume for the sake of contradiction that there exists a graph $G' \in \text{MBG}(G)$ so that $b(G') = b(\text{MBG}(G)) \leq 2$ and $Sb_2(G) < b(G) + b(G')$. Since $Sb_2(G) \neq b(G) + b(G')$, there exists a graph $G'' \in \text{BG}(G) - \text{MBG}(G)$ where G'' is the result of a bondage move on G of size y so that $Sb_2(G) = y + b(G'')$. Since $G'' \notin MBG(G)$ we know that $y \geq b(G) + 1$. Therefore, $Sb_2(G) \geq b(G) + 1 + b(G'')$, which leads to

$$b(G) + 1 + b(G'') \leq Sb_2(G) < b(G) + b(G'),$$

which implies

$$1 + b(G'') < b(G').$$

This yields a contradiction since $b(G'') \geq 1$, but we assumed that $b(G') \leq 2$. □

Theorem 6 together with the bondage number of a path graph from [5] as provided below, assists in determining 2-synchronous bondage of path graphs.

Theorem 7. *The bondage number of a path of order $n \geq 2$ is given by*

$$b(P_n) = \begin{cases} 2, & \text{if } n \equiv 1 (\bmod 3) \\ 1, & \text{otherwise}. \end{cases}$$

A *pendant edge* is an edge that is incident to a vertex of degree 1. Next, we must show that a pendant edge in P_n is always in a minimum bondage set.

Lemma 8. *For any path graph P_n of order $n \geq 2$ with pendant edge e, there exists a minimum bondage set which contains e.*

Proof. Given any path graph, P_m, of order $m \geq 1$, $\gamma(P_m) = \lceil \frac{m}{3} \rceil$ (see [5] for example). Let e_1 be a pendant edge in P_n. We proceed by cases in using Theorem 7 to determine $b(P_n)$.

Case 1. $n \equiv 0 (\bmod\ 3)$: Removing e_1 results in $P_1 \oplus P_{n-1}$. The domination number of the resulting disjoint graph is $\gamma(P_1 \oplus P_{n-1}) = \gamma(P_1) + \gamma(P_{n-1}) = 1 + \lceil \frac{n-1}{3} \rceil = 1 + \frac{n-1+1}{3} = 1 + \frac{n}{3} = 1 + \gamma(P_n)$. Since $b(P_n) = 1$ and $\gamma(P_n - \{e_1\}) = 1 + \gamma(P_n)$, the set $\{e_1\}$ is a minimum bondage set.

Case 2. $n \equiv 2 (\bmod\ 3)$: Removing e_1 results in $P_1 \oplus P_{n-1}$ whose domination number is $\gamma(P_1 \oplus P_{n-1}) = \gamma(P_1) + \gamma(P_{n-1}) = 1 + \lceil \frac{n-1}{3} \rceil = 1 + \frac{n-1+2}{3} = 1 + \frac{n+1}{3} = 1 + \gamma(P_n)$. Since $b(P_n) = 1$ and $\gamma(P_n - \{e_1\}) = 1 + \gamma(P_n)$, the set $\{e_1\}$ is a minimum bondage set.

Case 3. $n \equiv 1 (\bmod\ 3)$: For this case we know $n \geq 4$ and, therefore, P_n will have two pendant edges, denoted e_1 and e_2. Removing e_1 and e_2 results in $P_1 \oplus P_1 \oplus P_{n-2}$. The resulting domination number of this disjoint graph is $\gamma(P_1 \oplus P_1 \oplus P_{n-2}) = 2\gamma(P_1) + \gamma(P_{n-2}) = 2 + \lceil \frac{n-2}{3} \rceil = 1 + \frac{n+2}{3} = 1 + \gamma(P_n)$. Since $b(P_n) = 2$ and $\gamma(P_n - \{e_1, e_2\}) = 1 + \gamma(P_n)$, the set $\{e_1, e_2\}$ is a minimum bondage set. □

We now prove 2-synchronous bondage for all P_n.

Theorem 9. *For a path graph, P_n,*

$$Sb_2(P_n) = \begin{cases} 2, & n \equiv 0 (\bmod\ 3) \\ 3, & \text{otherwise}. \end{cases}$$

Proof. We know from Lemma 8 that each pendant edge in P_n is contained in a minimum bondage set.

Case 1. $n \equiv 0 (\bmod\ 3)$. Consider removing the two pendant edges. Let $P'_n = P_1 \oplus P_1 \oplus P_{n-2}$. So $\gamma(P'_n) = 2 + \lceil \frac{n-2}{3} \rceil = 2 + \frac{n}{3} = 2 + \gamma(P_n)$, and therefore $Sb_2(P_n) \leq 2$. By Proposition 5 we see $Sb_2(P_n) = 2$.

Case 2. $n \equiv 1 (\bmod\ 3)$. We will assume that $n > 1$ because if $n = 1$ we can not increase the domination number by 2. Note that $b(P_n) = 2$ by Theorem 7. Therefore, $Sb_2(P_n) \geq 3$ by Proposition 5. Consider removing three edges, $E = \{e_1, e_2, e_3\}$, so that $P_n - E = P_1 \oplus P_1 \oplus P_1 \oplus P_{n-3}$. Note that $\gamma(P_n - E) = 3 + \gamma(P_{n-3}) = 3 + \lceil \frac{n-3}{3} \rceil = 2 + \lceil \frac{n}{3} \rceil = 2 + \gamma(P_n)$ which implies $Sb_2(P_n) \leq 3$. Therefore, we can conclude that $Sb_2(P_n) = 3$.

Case 3. $n \equiv 2 (\bmod\ 3)$. We will assume that $n > 2$ because if $n = 2$ we can not increase the domination number by 2. Assume by contradiction that $Sb_2(P_n) = 2$.

By Proposition 5, there must be an edge e so that, when removed, we increase the domination number by 1. The removal of an edge from P_n will result in $P_a \oplus P_b$ where $a+b=n$. Since $n \equiv 2 (\text{mod } 3)$ then there are two possible options for the pair a,b; $a,b \equiv 1(\text{mod } 3)$ or, without loss of generality, $a \equiv 0(\text{mod } 3)$ and $b \equiv 2(\text{mod } 3)$. If $a \equiv 0(\text{mod } 3)$ and $b \equiv 2(\text{mod } 3)$ we know $\gamma(P_a \oplus P_b) = \gamma(P_a) + \gamma(P_b) = \frac{a}{3} + \lceil \frac{b}{3} \rceil = \frac{a}{3} + \frac{b+1}{3} = \frac{n+1}{3} = \lceil \frac{n}{3} \rceil$. Therefore $\gamma(P_a \oplus P_b) = \gamma(P_n)$ which tells us that the removal of the edge did not change the domination number. So $\text{MBG}(P_n) \subseteq \{P_a \oplus P_b : a,b \equiv 1(\text{mod } 3)\}$. By Theorem 7 we can conclude $b(\text{MBG}(P_n)) = 2$ since $a,b \equiv 1(\text{mod } 3)$. Since we know from Theorem 7 that $b(P_n) = 1$, Theorem 6 implies $Sb_2(P_n) = 3$. □

To conclude this section, we find bounds for general graphs based on specific characteristics including the degree of a vertex and induced subgraph structure. In [2] and [5], the authors show that the bondage number of a graph is bounded above by the minimum of one less than the sum of the degrees of two adjacent vertices. We can generalize their results to find an upper bound for $Sb_2(G)$ based on the degree of several vertices.

Theorem 10. *Let $G(V,E)$ be a graph. Then*

$$Sb_2(G) \leq min\{\deg(u) + \deg(v) + \deg(w) - \sigma(u,v,w)\},$$

where the minimum is over all sets $\{u,v,w\} \subseteq V$ where v is adjacent to both u and w, and $\sigma(u,v,w)$ is the number of edges in the induced subgraph on $\{u,v,w\}$.

Proof. Let $\{u,v,w\} \subseteq V$ be such that v is adjacent to both u and w. Let σ denote the size of the induced subgraph on u,v,w, and let $\lambda = \deg(u) + \deg(v) + \deg(w) - \sigma$. If E' is the set of edges incident to u,v, or w, then $|E'| = \lambda$.

Assume, for the sake of contradiction, that $Sb_2(G) > \lambda$. Therefore, if $G' = G - E'$, then u, v, and w are isolated in G' and $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. If D is a minimum dominating set of $G - \{u,v,w\}$, then $D \cup \{u,v,w\}$ is a minimum dominating set for G' and $D \cup \{v\}$ dominates G. Therefore $\gamma(G') = |D| + 3$ and $\gamma(G) \leq |D| + 1$ which contradicts that $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. □

Theorem 11. *Let $G(V,E)$ be a graph. Then*

$$Sb_2(G) \leq min\{\deg(u) + \deg(v) + \deg(s) + \deg(t) - 2\},$$

where the minimum is over all subsets $\{u,v,s,t\} \subseteq V$ where $uv, st \in E$ and the size of the induced subgraph on $\{u,v,s,t\}$ is two.

Proof. Let $\{u,v,s,t\} \subseteq V$ where $uv, st \in E$ and the size of the induced subgraph on $\{u,v,s,t\}$ is two. Let $\lambda = \deg(u) + \deg(v) + \deg(s) + \deg(t) - 2$. If E' is the set of edges incident to $u,v,s,$ or $,t$, then $|E'| = \lambda$.

Assume, for the sake of contradiction, that $Sb_2(G) > \lambda$. Therefore, if $G' = G - E'$, then $u,v,w,$ and t are isolated in G' and $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. If D is a minimum dominating set of $G - \{u,v,s,t\}$, then $D \cup \{u,v,s,t\}$ is a minimum dominating set for G' and $D \cup \{u,s\}$ dominates G. Therefore $\gamma(G') = |D| + 4$ and $\gamma(G) \leq |D| + 2$ which contradicts that $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. □

4 Properties of Sb_k and Application of Sb_k to Graph Families

If a graph has multiple components, the k-synchronous bondage number can be found by taking the minimum of the sum of the l_i-synchronous bondage number for a subset of i components where $\sum l_i = k$. We note the example where $k = 2$ in the following proposition. This can be easily generalized to larger values of k.

Proposition 12. *If a graph G consists of n components $C_1, C_2, \ldots, C_n$ where $1 \leq b(C_1) \leq b(C_2) \leq \ldots \leq b(C_n)$, then*

$$Sb_2(G) = min\{Sb_2(C_1), Sb_2(C_2), \ldots, Sb_2(C_n), b(C_1) + b(C_2)\}.$$

We can also easily find the k-synchronous bondage number of a graph if there are sufficiently many pendant edges. To do so, Lemma 13 gives a sufficient condition for a vertex to be in a minimum dominating set.

Lemma 13. *Let $G = (V, E)$ be a graph with vertices $r, a, b \in V$ so that $ra, rb \in E$ and $\deg(a) = \deg(b) = 1$ (i.e. ra and rb are pendant edges). Then any minimum dominating set must contain r.*

Proof. Let $A \subseteq V$ be any minimum dominating set of G, and assume for the sake of contradiction that $r \notin A$. Then, it must be true that $a, b \in A$ since these are adjacent to no other vertices. But observe that the set $A^{'} = (A \setminus \{a, b\}) \cup \{r\}$ must also be a dominating set and $|A'| < |A|$, contradicting the minimality of A. Thus, $r \in A$. □

If we have a graph that contains a vertex that is incident to more than one pendant edge, we can find its bondage number of such a graph as shown in the following theorem.

Theorem 14. *Let $G = (V, E)$ be a graph with vertices $r, a, b \in V$ such that $ra, rb \in E$ and $\deg(a) = \deg(b) = 1$ (that is, ra and rb are pendant edges). Then, $Sb_1(G) = 1$.*

Proof. Note by Proposition 5, $Sb_1(G) \geq 1$. Next, define $G' = G - ra$, and let $A \subseteq V$ be a minimum dominating set of G. By Lemma 13 above, we know that $r \in A$. The set $A^{'} = A \cup \{a\}$ is a dominating set in G', which we claim is a minimum dominating set. To prove this, suppose for the sake of contradiction that there exists some $B^{'} \subseteq V$ so that $|B^{'}| < |A^{'}| = |A| + 1$ which dominates G'. Clearly, a must be in $B^{'}$, so if we define $B = B^{'} - \{a\}$, this is a dominating set of G with $|B| < |A|$, which was a minimal dominating set of G. This is a contradiction, so we must have that $A^{'}$ is a minimal dominating set of G', so that the bondage number of G must be one. □

Applying Theorem 14 repeatedly to a graph G which contains sufficiently many pendants, along with Proposition 5 gives the immediate result.

Corollary 15. *Let $R \subseteq V$ be such that for all $r \in R$, r is incident to at least two pendant edges in E. If $A = \{a \in V : d(a) = 1$ and a is adjacent to a vertex in $R\}$, then $Sb_{|A|-|R|}(G) = |A| - |R|$.*

Now we move on to prove the k-synchronous bondage for several well-known graph families: paths, cycles, trees, and complete graphs.

Theorem 16. *For a path graph, P_n,*

$$Sb_k(P_n) = \begin{cases} \lfloor \frac{3k-1}{2} \rfloor, & n \equiv 0 (\text{mod } 3) \\ \lfloor \frac{3k+1}{2} \rfloor, & n \equiv 1 (\text{mod } 3) \\ \lceil \frac{3k-1}{2} \rceil, & n \equiv 2 (\text{mod } 3). \end{cases}$$

Proof. Note that if we remove a set $\mathscr{E} \subseteq E$ from P_n then $P_n - \mathscr{E} = \bigoplus_{i=1}^{|\mathscr{E}|} P_{a_i}$, where $\sum_{i=1}^{|\mathscr{E}|+1} a_i = n$. This then gives us

$$\gamma(P_n - \mathscr{E}) \leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}|+1) \tag{1}$$

because

$$\begin{aligned} \gamma(P_n - \mathscr{E}) &= \sum_{i=1}^{|\mathscr{E}|+1} \gamma(P_{a_i}) \\ &= \sum_{i=1}^{|\mathscr{E}|+1} \left\lceil \frac{a_i}{3} \right\rceil \\ &= \sum_{a_i \equiv 0 (\text{mod } 3)} \frac{a_i}{3} + \sum_{a_i \equiv 1 (\text{mod } 3)} \frac{a_i+2}{3} + \sum_{a_i \equiv 2 (\text{mod } 3)} \frac{a_i+1}{3} \\ &= \sum_{i=1}^{|\mathscr{E}|+1} \frac{a_i}{3} + \sum_{a_i \equiv 1 (\text{mod } 3)} \frac{2}{3} + \sum_{a_i \equiv 2 (\text{mod } 3)} \frac{1}{3} \\ &\leq \frac{n}{3} + \sum_{a_i \equiv 1 (\text{mod } 3)} \frac{2}{3} + \sum_{a_i \equiv 2 (\text{mod } 3)} \frac{2}{3} \\ &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}|+1). \end{aligned}$$

We will proceed by cases.

Case 1: $n \equiv 0 (\text{mod } 3)$. First, we will show there exists a set of $\lfloor \frac{3k-1}{2} \rfloor$ edges, E', so that $\gamma(P_n - E') = \gamma(P_n) + k$. Let E' consists of the leftmost $\lfloor \frac{3k-1}{2} \rfloor$ edges of P_n. For simplicity, let $y = \lfloor \frac{3k-1}{2} \rfloor$. Thus, $P_n - E' = P_{n-y} \oplus \left(\bigoplus_{i=1}^{y} P_1 \right)$. So the domination number of $P_n - E'$ is

$$\gamma(P_n - E') = \left\lceil \frac{n-y}{3} \right\rceil + y = \left\lceil \frac{n+2y}{3} \right\rceil.$$

Consider that when k is odd, $y = \frac{3k-1}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k - \frac{1}{3} \rceil = \frac{n}{3} + k$. And when k is even, $y = \frac{3k-1}{2} - \frac{1}{2} = \frac{3k-2}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k - \frac{2}{3} \rceil = \frac{n}{3} + k$. These equalities imply that $\gamma(P_n - E') = \frac{n}{3} + k = \gamma(P_n) + k$.

Now we move to show $\lfloor \frac{3k-1}{2} \rfloor$ is the minimum number of edges which increase $\gamma(P_n)$ by k. Assume for the sake of contradiction that $Sb_k(P_n) < \lfloor \frac{3k-1}{2} \rfloor$. This implies that there is a set $\mathscr{E} \subseteq E$ such that $|\mathscr{E}| < \lfloor \frac{3k-1}{2} \rfloor$ and $\gamma(P_n - \mathscr{E}) \geq \lceil \frac{n}{3} \rceil + k = \frac{n}{3} + k$. By equation (1) we see

$$\begin{aligned}\gamma(P_n - \mathscr{E}) &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}| + 1)\\ &\leq \frac{n}{3} + \frac{2}{3}\left(\left\lfloor \frac{3k-1}{2} \right\rfloor\right)\\ &\leq \frac{n}{3} + \frac{2}{3}\left(\frac{3k-1}{2}\right)\\ &= \frac{n}{3} + k - \frac{1}{3}.\end{aligned}$$

But this contradicts that $\gamma(P_n - \mathscr{E}) \geq \frac{n}{3} + k$. Therefore, we have proven that $Sb_k(P_n) = \lfloor \frac{3k-1}{2} \rfloor$ for $n \equiv 0 (\text{mod } 3)$.

Case 2: $n \equiv 1 (\text{mod } 3)$. Similar to the previous case, let E' consists of the leftmost $\lfloor \frac{3k+1}{2} \rfloor$ edges of P_n and let $y = \lfloor \frac{3k+1}{2} \rfloor$. Thus, $P_n - E' = P_{n-y} \oplus \left(\bigoplus_{i=1}^{y} P_1 \right)$. So the domination number of $P_n - E'$ is

$$\gamma(P_n - E') = \left\lceil \frac{n-y}{3} \right\rceil + y = \left\lceil \frac{n+2y}{3} \right\rceil.$$

When k is odd, $y = \frac{3k+1}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k + \frac{1}{3} \rceil = \frac{n+2}{3} + k$. When k is even, $y = \frac{3k}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k \rceil = \frac{n+2}{3} + k$. These equalities imply that $\gamma(P_n - E') = \frac{n+2}{3} + k = \gamma(P_n) + k$.

Now we will show $\lfloor \frac{3k+1}{2} \rfloor$ is the minimum number of edges which increase $\gamma(P_n)$ by k. Assume for the sake of contradiction that $Sb_k(P_n) < \lfloor \frac{3k+1}{2} \rfloor$. This implies that there is a set $\mathscr{E} \subseteq E$ such that $|\mathscr{E}| < \lfloor \frac{3k+1}{2} \rfloor$ and $\gamma(P_n - \mathscr{E}) \geq \lceil \frac{n}{3} \rceil + k = \frac{n+2}{3} + k$. By equation (1) we see

$$\begin{aligned}\gamma(P_n - \mathscr{E}) &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}| + 1)\\ &\leq \frac{n}{3} + \frac{2}{3}\left(\left\lfloor \frac{3k+1}{2} \right\rfloor\right)\\ &\leq \frac{n}{3} + k + \frac{1}{3}.\end{aligned}$$

But this contradicts that $\gamma(P_n - \mathscr{E}) \geq \frac{n+2}{3} + k$. Therefore, we have proven that $Sb_k(P_n) = \lfloor \frac{3k+1}{2} \rfloor$ for $n \equiv 1 (\text{mod } 3)$.

Case 3: $n \equiv 2 (\text{mod } 3)$. As before, let E' consists of the leftmost $\lceil \frac{3k-1}{2} \rceil$ edges of P_n and let $y = \lceil \frac{3k-1}{2} \rceil$. Thus, $P_n - E' = P_{n-y} \oplus \left(\bigoplus_{i=1}^{y} P_1 \right)$. So the domination number of $P_n - E'$ is

$$\gamma(P_n - E') = \left\lceil \frac{n-y}{3} \right\rceil + y = \left\lceil \frac{n+2y}{3} \right\rceil.$$

When k is odd, $y = \frac{3k-1}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k - \frac{1}{3} \rceil = \frac{n+1}{3} + k$. When k is even, $y = \frac{3k}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k \rceil = \frac{n+1}{3} + k$. These equalities imply that $\gamma(P_n - E') = \frac{n+1}{3} + k = \gamma(P_n) + k$.

Now we will show $\lceil \frac{3k-1}{2} \rceil$ is the minimum number of edges which increase $\gamma(P_n)$ by k. Assume for the sake of contradiction that $Sb_k(P_n) < \lceil \frac{3k-1}{2} \rceil$. This implies that there is a set $\mathscr{E} \subseteq E$ such that $|\mathscr{E}| < \lceil \frac{3k-1}{2} \rceil$ and $\gamma(P_n - \mathscr{E}) \geq \lceil \frac{n}{3} \rceil + k = \frac{n+1}{3} + k$. By equation (1) we see

$$\begin{aligned}\gamma(P_n - \mathscr{E}) &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}| + 1) \\ &\leq \frac{n}{3} + \frac{2}{3}\left(\left\lceil \frac{3k-1}{2} \right\rceil \right) \\ &\leq \frac{n}{3} + \frac{2}{3}\left(\frac{3k}{2} \right) \\ &= \frac{n}{3} + k.\end{aligned}$$

But this contradicts that $\gamma(P_n - \mathscr{E}) \geq \frac{n+1}{3} + k$. Therefore, we have proven that $Sb_k(P_n) = \lceil \frac{3k-1}{2} \rceil$ for $n \equiv 2 (\text{mod } 3)$. □

It can be easily verified that for a cycle of order $n \geq 3$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ (see [5] for example). Since removing any single edge from C_n results in P_n and $\gamma(C_n) = \gamma(P_n)$ the removal of any single edge of a cycle does not change the cycle's domination number. Therefore we have the following theorem.

Theorem 17. *For a cycle graph, C_n,*

$$Sb_k(C_n) = \begin{cases} \lfloor \frac{3k-1}{2} \rfloor + 1, & n \equiv 0 (\text{mod } 3) \\ \lfloor \frac{3k+1}{2} \rfloor + 1, & n \equiv 1 (\text{mod } 3) \\ \lceil \frac{3k-1}{2} \rceil + 1, & n \equiv 2 (\text{mod } 3). \end{cases}$$

For the bondage number of trees, [5] established the following upper bound.

Theorem 18. *If T is a nontrivial tree, then $b(T) \leq 2$.*

We can extend Theorem 18 to provide a range of values for Sb_k, and these values are sharp.

Corollary 19. *Given a tree, T, with at least k edges, then $k \leq Sb_k(T) \leq 2k$ and the bounds are sharp.*

Proof. The bounds follow immediately from Proposition 5 and repeated iterations of Theorem 18. The lower bound is sharp if we consider a star graph with k edges, whose domination number is 1. To increase the domination number by k we must remove all k edges. To show that the upper bound is also sharp, we will define a special spider graph, S_k^*, as a rooted tree with $2k+2$ vertices so that the root vertex will have k children of degree 2 and one child of degree 1. Notice Figure 2 is S_2^*. Notice that $|V(S_k^*)| = 2k+2$, $|E(S_k^*)| = 2k+1$, and $\gamma(S_k^*) = k+1$. If we want to find $Sb_k(S_k^*)$ we must produce a graph that has a domination number of $2k+1$. For a graph of order n to have a domination number of $n-1$, the graph must have only 1 edge. Therefore $Sb_k(S_k^*) = 2k+1-1 = 2k$. □

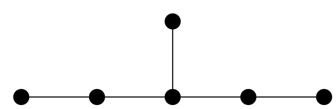


Figure 2: $Sb_2(T) = 4$.

The result for $Sb_k(K_n)$ for complete graphs stems from the following theorem from [11] which gives an upper bound for the number of edges in a graph with a specific domination number.

Theorem 20. *If G is a graph of order n and $2 \le \gamma(G) \le n$, then the number of edges of G is at most $\left\lfloor \frac{(n-\gamma(G))(n-\gamma(G)+2)}{2} \right\rfloor$. Equality occurs if and only if G is the disjoint union of $\gamma(G)-2$ isolated vertices and a graph obtained by removing from an $(n-\gamma(G)+2)$-clique the edges belonging to a minimum covering.*

Using Theorem 20, we present the following corollary.

Corollary 21. *For any complete graph, K_n,*

$$Sb_k(K_n) = \binom{n}{2} - \left\lfloor \frac{(n-k-1)(n-k+1)}{2} \right\rfloor.$$

Proof. From [11], we know that for any simple graph of order n and domination number d, the maximum number of edges that the graph can have is

$$\left\lfloor \frac{(n-d)(n-d+2)}{2} \right\rfloor.$$

Clearly this graph must be a subgraph of K_n. Note also that K_n has $\binom{n}{2}$ edges and $\gamma(K_n) = 1$, and so we can remove

$$\binom{n}{2} - \left\lfloor \frac{(n-k-1)(n-k+1)}{2} \right\rfloor$$

edges to leave only the subgraph mentioned in Theorem 20 with domination number $k+1$ for any positive integer k. The quantity above must be the minimum number of edges we must remove to increase the domination number of K_n by k, because if there were any smaller edge set E whose removal would increase the domination number by k, then the resulting graph $K_n - E$ would contradict Theorem 20. Thus,

$$Sb_k(K_n) = \binom{n}{2} - \left\lfloor \frac{(n-k-1)(n-k+1)}{2} \right\rfloor.$$

□

5 Conclusion

The study of dominations numbers has many applications including network design (see for example [1] and [9]). The study of bondage numbers and Sb_k are particularly important to understand the vulnerability of these networks. In this paper, we have considered applying the idea of Sb_k to sparse graphs such as paths, cycles, and trees and the extremely dense complete graph. Investigating Sb_k for additional graph families such as grid graphs or r-regular graphs would help us further understand the vulnerability of particular network structures. The interaction between Sb_k and graph operators, including disjoint unions, is another area to consider.

Furthermore, in order to create sharp bounds without regard to graph families, we must delve into the $G(n,m)$ problem; that is, given any graph on n vertices with m edges, what is the maximum number of edges that we might have to remove in order to cause a failure state? In this way, we avoid being restricted by whether a particular graph belongs to a graph family for which Sb_k has been previously determined. If we could state for certain that the removal of e edges would guarantee a failure state, we could turn our focus to efficiently determining which e edges need to be removed.

Our presentation of specific Sb_2 properties and Sb_k in general aims to extend the idea of a bondage number and thereby provide a new criteria for a failure state in a network. Since, in Section 3, we demonstrated that it is possible for a 2-synchronous bondage move to be more effective than the successive one-step counterparts, further study of Sb_k is worthwhile. To better evaluate the benefits of studying Sb_2, we would need to establish how much more efficient a 2-synchronous bondage move can be. Similarly, we would like to determine the increase in efficiency achieved by k-synchronous bondage moves when compared to successive n-synchronous bondage moves where $n < k$ and n is a factor of k.

Funding

This research was done at Moravian University as part of the Computational and Experimental Mathematics REU program; it was funded by the National Science Foundation (NSF Grant DMS-1852378).

Bibliography

[1] A. Arora and K. Factor. Domination and network stability: an application. volume 175, pages 9–19. 2005. 36th Southeastern International Conference on Combinatorics, Graph Theory, and Computing.

[2] D. Bauer, F. Harary, J. Nieminen, and C. Suffel. Domination alteration sets in graphs. *Discrete Mathematics*, 47:153–161, 1983.

[3] L. Beineke and F. Harary. The connectivity function of a graph. *Mathematika*, 14(2):197–202, 1967.

[4] F. Boesch, D. Gross, W. Kazmierczak, C. Suffel, and A. Suhartomo. Component order edge connectivity - an introduction. *Proceedings of the Thirty-Seventh*

Southeastern International Conference on Combinatorics, Graph Theory and Computing - Conger. Numen., 178:7–14, 2006.

[5] J.F. Fink, M. Jacobson, L. Kinch, and J. Roberts. The bondage number of a graph. *Discrete Mathematics*, 86(1-3):47–57, 1990.

[6] D. Gross, M. Heinig, L. Iswara, W. Kazmierczak, K. Luttrell, J. Saccoman, and C. Suffel. A survey of component order connectivity models of graph theoretic networks. *WSEAS Transactions on Mathematics*, 12:895–910, September 2013.

[7] D. Gross, M. Heinig, J. Saccoman, and C. Suffel. On neighbor component order edge connectivity. *Congressus Numerantium*, 223:17 – 32, 01 2015.

[8] T. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of Domination in Graphs.* Marcel Dekker, New York, 1998.

[9] F. Roberts. *Graph theory and its applications to problems of society*, volume 29 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1978.

[10] E. Sampathkumar. Connectivity of a graph-a generalization. *J. Comb. Inf. Syst. Sci*, 9(2):71–78, 1984.

[11] V. G. Vizing. An estimate of the external stability number of a graph. *Dokl. Akad. Nauk SSSR*, 164:729–731, 1965.

[12] D. West. *Introduction to Graph Theory*. Prentice Hall, Upper Saddle River, NJ 07458, 2 edition, 2001.

[13] J. Xu. On bondage numbers of graphs: A survey with some comments. *International Journal of Combinatorics*, 2013.