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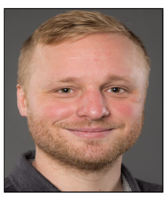
Characterizing distances between points in the level sets of a class of continuous functions on a closed interval

Henry Riely, Yuanming Luo*



Henry Riely received his Ph.D. from the Washington State University in 2019. He is a lecturer at Kennesaw State University. His main mathematical interests lie in analysis, especially stochastic processes and harmonic analysis.

Yuanming Luo is a junior undergraduate student at Georgia Institute of Technology. He is working toward both math and computer science degrees. His primary research interests are partial differential equations and numerical methods..

Abstract

Given a continuous function $f : [a,b] \to \mathbb{R}$ such that $f(a) = f(b)$, we investigate the set of distances $|x-y|$ where $f(x) = f(y)$. In particular, we show that the only distances this set must contain are ones which evenly divide $[a,b]$. Additionally, we show that it must contain at least one third of the interval $[0, b-a]$. Lastly, we explore some higher dimensional generalizations.

1 Introduction

Imagine waking up on a crisp fall morning and deciding to use the day for a hike. You drive to the trailhead and begin to chart a route. Since you must return to your car, your elevation will be the same at the beginning and the end of your walk. Are there other elevations through which you will pass twice? Clearly there are. If you

***Corresponding author:** hriely@kennesaw.edu

begin on an ascent, you must descend to return to the trailhead, and if you begin with a decent, you must eventually ascend. If the trail is perfectly flat, then at every moment your elevation is shared by every other moment. This intuition is often given in an introductory calculus course to illustrate the intermediate value theorem.

A natural follow up question: Can anything be said about the time elapsed between two points of equal elevation? For instance, if your hike lasts an hour, we know that there are two instants, separated by an hour, of equal elevation, namely the start and the finish. Need there be two such instances separated by a half hour? The answer, it turns out, is yes. Separated by 25 minutes? No, it's possible to design a hike with no 25 minute time interval leaving you at the same elevation that you started. So what is special about 30 minutes? Can we characterize all such durations? In this paper, we answer this and related questions.

More abstractly, we will investigate the level sets of real-valued continuous functions on closed intervals, whose endpoints get sent to the same real number. To map these functions onto our hiking analogy, given $f : [a,b] \to \mathbb{R}$, we can think of a and b as the start and end times of our hike, and we can think of $f(x)$ as our elevation at time x. Then f must be continuous to rule out teleportation and $f(a) = f(b)$ so that we begin and end at the same elevation. Each level set of f can be thought of as a collection of times of equal elevation (we will call these times *isopoints*). In this paper we will study the distances between all isopoints. In our framing, the distance between isopoints should be thought of as a time-distance; however, thinking of a and b as locations in space is an equally valid interpretation.

In Theorem 1, we will show that every duration of time that evenly divides the total length of the hike is a distance between isopoints. In Theorem 2, we will prove that that Theorem 1 gives the *only* such distances common to all possible hikes. Lastly, we will prove a sequence of Lemmas to build up to Theorem 3, our main result, which states that at least one third of the time-lengths between 0 and the total duration of the hike are distances between isopoints.

2 Notation

Given a nonempty closed interval $[a,b] \subset \mathbb{R}$ and a real number λ, we will use $C_\lambda([a,b])$ to represent the set of continuous functions on $[a,b]$ mapping both endpoints to λ. More precisely,

$$C_\lambda([a,b]) := \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous and } f(a) = f(b) = \lambda\}$$

We will use $C_\mathbb{R}([a,b])$ to refer to all functions in some $C_\lambda([a,b])$:

$$C_\mathbb{R}([a,b]) := \bigcup_{\lambda \in \mathbb{R}} C_\lambda([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous and } f(a) = f(b)\}$$

Given a function $f \in C_\mathbb{R}([a,b])$, and a subset $X \subseteq [a,b]$, let

$$D_f(X) := \{d > 0 : |x - y| = d \text{ and } f(x) = f(y) \text{ for some } x, y \in X\}$$

If X is absent as in D_f, assume $X = [a,b]$.

The contents of this paper are motivated by Exercise 5.4.6. in [1].

A^{o}, $\overline{A}$, and ∂A will be used to denote the topological interior, closure, and boundary of A respectively. $\mu(A)$ will be used for the Lebesgue measure of A.

3 Main Results

We begin by proving that, for every $f \in C_{\mathbb{R}}([a,b])$, D_f contains the sequence $b-a$, $\frac{b-a}{2}$, $\frac{b-a}{3}$, $\ldots$

Theorem 1. *Let f be a real-valued, continuous function on the closed interval $[a,b]$ such that $f(a) = f(b)$. Given any $n \in \mathbb{N}$, there exist x and y in $[a,b]$ such that $|x-y| = \frac{b-a}{n}$ and $f(x) = f(y)$.*

Proof. We may assume without loss of generality that $[a,b] = [0,1]$. If not, just apply the result to $f(a+(b-a)x)$.

Define $g(x) = f(x+\frac{1}{n}) - f(x)$ and consider the sum

$$g(0) + g\left(\frac{1}{n}\right) + g\left(\frac{2}{n}\right) + \cdots + g\left(\frac{n-1}{n}\right) \tag{1}$$

$$= f\left(\frac{1}{n}\right) - f(0) + f\left(\frac{2}{n}\right) - f\left(\frac{1}{n}\right) + \cdots + f(1) - f\left(\frac{n-1}{n}\right) \tag{2}$$

$$= f(1) - f(0) = 0 \tag{3}$$

where (3) follows from (2) due to cancellation.

If every term in (1) is 0, then the result follows immediately because $f(\frac{k+1}{n}) = f(\frac{k}{n})$ for $k = 0, 1, \ldots, n-1$. If (1) contains one or more nonzero terms, then there must be at least one positive and one negative term in order for the sum to be zero. That is, $g\left(\frac{k_1}{n}\right) < 0$ and $g\left(\frac{k_2}{n}\right) > 0$ for some integers k_1 and k_2 between 0 and $n-1$. Thus, by the intermediate value theorem, $g(c) = 0$ for some c between $\frac{k_1}{n}$ and $\frac{k_2}{n}$ (the continuity of g follows from the continuity of f). Therefore, we have $f(c+\frac{1}{n}) - f(c) = 0$. □

Theorem 1 provides a partial answer to the question posed in the introduction. If we hike for an hour, there will be two instants, 30 minutes apart, of equal elevation because 30 minutes is half of an hour. The same is true for 20 minutes, 15 minutes, etc. We are not done, however, because we haven't ruled out other durations. Our next result shows that no other duration is *guaranteed* to separated two equipoints.

Theorem 2. *Given a closed interval $[a,b]$, let $0 < d < b-a$. If d is not of the form $\frac{b-a}{n}$ for some $n \in \mathbb{N}$, then there exists a continuous function $f : [a,b] \to \mathbb{R}$ with $f(a) = f(b)$ such that $d \notin D_f$.*

Proof. Once again, we can assume without loss of generality that $[a,b] = [0,1]$. First, let $p(x)$ be any continuous d-periodic function with $p(0) \neq p(1)$. Note that the existence of such functions hinges on the fact that $d \neq \frac{1}{n}$. Next, let $m(x)$ be any strictly monotone continuous function such that $m(0) = 0$ and $m(1) = p(0) - p(1)$. We can insist on strict monotonicity since $m(0) = 0 \neq p(0) - p(1) = m(1)$. Then $p+m$ is

The definitions of $C_{\mathbb{R}}([a,b])$ and D_f are given in Section 2: Notation.

continuous as the sum of continuous functions. Furthermore, $(p+m)(0) = p(0) = p(1) + p(0) - p(1) = (p+m)(1)$.

To finish, we must show that $d \notin D_{p+m}$. Indeed, for all $x \in [0, 1-d]$, we have

$$\begin{aligned}(p+m)(x+d) - (p+m)(x) &= p(x+d) - p(x) + m(x+d) - m(x) \\ &= 0 + m(x+d) - m(x) \neq 0\end{aligned}$$

using the monotonicity of m and the periodicity of p. □

Taken together, Theorem 1 and Theorem 2 tell us that, on a hike that begins and ends at the same height, the only durations we know, a priori, will separate times of equal elevation, must evenly divide that total time of the hike. This is expressed formally in the following corollary:

Corollary 1.

$$\bigcap_{f \in C_{\mathbb{R}}([a,b])} D_f = \left\{ \frac{b-a}{n} : n \in \mathbb{N} \right\}.$$

Proof. Theorem 1 gives one inclusion and Theorem 2 gives the other. □

Corollary 1 characterizes the distances which are common to all functions in $C_{\mathbb{R}}([a,b])$. One then might wonder whether this represents a small intersection of large overlapping sets or there is a particular $f \in C_{\mathbb{R}}([a,b])$ such that $D_f = \left\{ \frac{b-a}{n} : n \in \mathbb{N} \right\}$. It turns out to be the former. Each D_f is considerably larger than the set of divisors of $b-a$. In fact, we show in Theorem 3 that each D_f contains at least a third of the numbers between 0 and $b-a$. Before we prove it, we need to develop a series of lemmas about D_f. We will start with results about the size of D_f for very simple functions, and generalize until we can analyze D_f for arbitrary $f \in C_{\mathbb{R}}([a,b])$. We begin by showing that shrinking the domain of f shrinks D_f.

Lemma 1. *If $A \subseteq B$, then $D_f(A) \subseteq D_f(B)$.*

Proof. Assume $d \in D_f(A)$. Then there are points $x, y \in A$ such that $|x-y| = d$ and $f(x) = f(y)$. But $A \subseteq B$, so x and y are also in B. Thus, $d \in D_f(B)$. □

Next, we will show that for constant functions f, D_f is at least as big as the domain of f.

Lemma 2. *Let f be a constant function on a bounded set $A \subset \mathbb{R}$. Assume A has a maximum value m. Then $\mu(D_f(A)) \geq \mu(A)$.*

Proof. Notice that $D_f(A)$ contains the set $m - A = \{m - a \mid a \in A\}$. Therefore $\mu(D_f(A)) \geq \mu(m-A) = \mu(A)$. □

In subsequent lemmas, it will be convenient to make assumptions like $f(x) > \lambda$ for all $x \in A$ or $\max_{A_1}(f) \leq \max_{A_2}(f)$. To help ensure we don't lose any generality, we will prove that certain transformations of f preserve D_f. More precisely, we will prove that D_f is invariant with repsect to horizontal and vertical reflections and translations of the graph of f. Since it's no extra work, we will prove a more general fact: that applying injective functions to the range of f and isometric functions to the domain of f does not affect D_f.

Lemma 3. *Let f be any real valued function on $A \subseteq \mathbb{R}$. If $g : f(A) \to \mathbb{R}$ is injective and $T : A \to \mathbb{R}$ is isometric, then $D_f(A) = D_{g\circ f}(A) = D_{f\circ T^{-1}}(T(A))$.*

Proof. Since g is injective, $f(x) = f(y)$ if and only if $g(f(x)) = g(f(y))$. Hence $d \in D_f(A)$ if and only if $d \in D_{g\circ f}(A)$, and so $D_f(A) = D_{g\circ f}(A)$.

Since all isometries are invertible, we have $f(x) = f(y)$ if and only if $f(T^{-1}Tx) = f(T^{-1}Ty)$. Furthermore, $|Tx-Ty| = |x-y|$ because T is isometric. Therefore, given any $d \geq 0$ there exist points $x, y \in A$ such that $d = |x-y|$ and $f(x) = f(y)$ if and only if there exists points $x', y' \in T(A)$ such that $f(T^{-1}x') = f(T^{-1}y')$ and $d = |x'-y'|$. Indeed, this correspondance is given by $x' = Tx$ and $y' = Ty$. Therefore, $D_f(A) = D_{f\circ T^{-1}}(T(A))$. □

In the next lemma, we will consider the case of continuous functions $f : [a,b] \to \mathbb{R}$ where $f(a)$ and $f(b)$ are both either global minima or global maxima. In other words, we will look at functions $f \in C_\lambda([a,b])$ where a and b are the *only* points where $f = \lambda$. It becomes quite easy to calculate D_f in this case.

Lemma 4. *Let $f \in C_\lambda([a,b])$ and suppose either $f(x) > \lambda$ for all $a < x < b$ or $f(x) < \lambda$ for all $a < x < b$. Then $D_f([a,b]) = (0, b-a]$.*

Proof. We may assume without loss of generality that $f(x) > \lambda$ for all $a < x < b$ because $D_f([a,b])$ does not change when the graph of f is reflected over the line $y = \lambda$, i.e., $D_f([a,b]) = D_{-f+2\lambda}([a,b])$, as we established in Lemma 3.

It is clear that $b-a \in D_f([a,b])$ since $f(a) = f(b)$, so we will let $d \in (0, b-a)$ and show that $d \in D_f([a,b])$. Define $g(x) = f(x+d) - f(x)$. Note that $g(a) = f(a+d) - f(a) = f(a+d) - \lambda > 0$ because $f(a+d) > \lambda$. Also, $g(b-d) = f(b) - f(b-d) = \lambda - f(b-d) < 0$ because $f(b-d) > \lambda$.

The intermediate value theorem guarantees the existence of a $c \in (a, b-d)$ such that $g(c) = f(c+d) - f(c) = 0$, i.e., $f(c+d) = f(c)$. Therefore $d \in D_f([a,b])$. □

Having settled the case where the global minima or maxima of f are located at the endpoints of a single closed interval, we will now ask the same question when f is defined on the union of two intervals. Once again, assuming that f has global minima at every endpoint point or global maxima at every endpoint, what does D_f look like? By Lemma 4, we already know how each interval will contribute to D_f when considered separately. In the following lemma, we characterize the "interactions" between the two intervals. For convenience, we will assume that every endpoint is the location of a global minimum and we will make an assumption about where the global maximum is located. After proving the lemma, we will discuss how those assumptions can be discarded using Lemma 3.

Lemma 5. *Given any $a_1 < a_2 \leq a_3 < a_4$, define $A = [a_1, a_2] \cup [a_3, a_4]$, and let $f : A \to \mathbb{R}$ be a continuous function such that $f(a_k) = \lambda$ for $1 \leq k \leq 4$. If $f(x) > \lambda$ for all $x \in A^{\circ}$ and $\max_{[a_1,a_2]}(f) \geq \max_{[a_3,a_4]}(f)$, then $D_f(A) \supseteq [a_3 - a_1, a_4 - a_1]$.*

Proof. It is clear that $a_3 - a_1, a_4 - a_1 \in D_f(A)$ since $f(a_1) = f(a_3) = f(a_4)$, so we will let $d \in (a_3 - a_1, a_4 - a_1)$ and show that $d \in D_f(A)$. We will do this in three cases, depending on whether d is greater than, less than, or equal to $a_4 - a_2$. Define $g(x) = f(x+d) - f(x)$.

Case 1: $d > a_4 - a_2$.

In this case, we compute $g(a_1) = f(a_1+d) - f(a_1) = f(a_1+d) - \lambda > 0$ and $g(a_4 - d) = f(a_4) - f(a_4-d) = \lambda - f(a_4-d) < 0$. Here, we've used that $a_1 + d \in (a_3, a_4)$ and $a_4 - d \in (a_1, a_2)$ and $f > \lambda$ on these two open intervals. The intermediate value theorem then guarantees a $c \in (a_1, a_4 - d)$ such that $g(c) = f(c+d) - f(c) = 0$. Hence $d \in D_f(A)$.

Case 2: $d < a_4 - a_2$.

In this case, once again we compute $g(a_1) = f(a_1+d) - f(a_1) = f(a_1+d) - \lambda > 0$. This time, however, we observe that $g(t) \leq 0$ for some $t \in (a_1, a_2)$. Otherwise, we would have $f(t+d) > f(t)$ for all $t \in (a_1, a_2)$, contradicting the assumption $\max_{[a_1,a_2]}(f) \geq \max_{[a_3,a_4]}(f)$.

If $g(t) = 0$, we have $f(t+d) = f(t)$. If $g(t) < 0$, then the intermediate value theorem gives a $c \in (t, a_2)$ such that $g(c) = f(c+d) - f(c) = 0$. In either case, $d \in D_f(A)$.

Case 3: $d = a_4 - a_2$.

This case is trivial as $f(a_2) = \lambda = f(a_4) = f(a_2 + d)$. □

The hypotheses we've inserted into Lemma 5 impose significant constraints on the scope of the result, so it's worth pausing to consider how these can be relaxed, beginning with the assumption that $f(x) > \lambda$ for all $x \in A^{\text{o}}$. It's not so simple as stating that D_f is invariant with respect to vertical reflections since the inequality $\max_{[a_1,a_2]}(f) \geq \max_{[a_3,a_4]}(f)$ becomes $\min_{[a_1,a_2]}(-f) \leq \min_{[a_3,a_4]}(-f)$. However, as a corollary to Lemma 5, we will show that we can combine the two cases by insisting that $\max_{[a_1,a_2]}(|f-\lambda|) \geq \max_{[a_3,a_4]}(|f-\lambda|)$.

Corollary 2. *Given any $a_1 < a_2 \leq a_3 < a_4$, define $A = [a_1,a_2] \cup [a_3,a_4]$, and let $f : A \to \mathbb{R}$ be a continuous function such that $f(a_k) = \lambda$ for $1 \leq k \leq 4$. Suppose either $f(x) > \lambda$ for all $x \in A^{\text{o}}$ or $f(x) < \lambda$ for all $x \in A^{\text{o}}$. If $\max_{[a_1,a_2]}(|f-\lambda|) \geq \max_{[a_3,a_4]}(|f-\lambda|)$, then $D_f(A) \supseteq [a_3 - a_1, a_4 - a_1]$.*

Proof. If $f(x) > \lambda$ for all $x \in A^{\text{o}}$ then $|f - \lambda| = f - \lambda$, and adding λ to both sides of $\max_{[a_1,a_2]}(f-\lambda) \geq \max_{[a_3,a_4]}(f-\lambda)$ gives $\max_{[a_1,a_2]}(f) \geq \max_{[a_3,a_4]}(f)$, so $D_f(A) \supseteq [a_3 - a_1, a_4 - a_1]$ by Lemma 5.

On the other hand, if $f(x) < \lambda$ for all $x \in A^{\text{o}}$ then $|f - \lambda| = -f + \lambda$ and subtracting λ from both sides of $\max_{[a_1,a_2]}(-f+\lambda) \geq \max_{[a_3,a_4]}(-f+\lambda)$ gives $\max_{[a_1,a_2]}(-f) \geq \max_{[a_3,a_4]}(-f)$. Applying Lemma 5 to $-f$ and invoking the invariance proven in Lemma 3, we have $D_f(A) = D_{-f}(A) \supseteq [a_3 - a_1, a_4 - a_1]$. □

We cannot easily discard the $\max_{[a_1,a_2]}(|f-\lambda|) \geq \max_{[a_3,a_4]}(|f-\lambda|)$ hypothesis of Corollary 2. It's tempting to say that we do not lose generality due to Lemma 3, however, we must be careful. Indeed, Lemma 3 says that $D_f(A) = D_{f\circ T^{-1}}(T(A))$ for all isometries $T : A \to \mathbb{R}$. However, given $f : [a_1,a_2] \cup [a_3,a_4] \to \mathbb{R}$ such that $\max_{[a_1,a_2]}(|f-\lambda|) < \max_{[a_3,a_4]}(|f-\lambda|)$, there is no isometry T mapping $[a_1,a_2] \cup [a_3,a_4]$ to itself such that $\max_{[a_1,a_2]}(|f\circ T^{-1}-\lambda|) \geq \max_{[a_3,a_4]}(|f\circ T^{-1}-\lambda|)$ unless $[a_1,a_2]$ and $[a_3,a_4]$ are the same length.

Now that we've studied D_f for functions defined on a single interval and the union of two intervals, we will generalize to functions defined on n intervals. Locating specific intervals becomes very complicated due to interactions among the intervals, so we will return to our goal of lower bounding the size of D_f.

Lemma 6. *Given any $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$, define $A = \cup_{k=1}^{n}[a_k,b_k]$ and let $f : A \to \mathbb{R}$ be a continuous function such that $f(a_k) = f(b_k) = \lambda$ for $1 \leq k \leq n$. Suppose either $f(x) > \lambda$ for all $x \in A^{\circ}$ or $f(x) < \lambda$ for all $x \in A^{\circ}$. Then $\mu(D_f(A)) \geq \mu(A)$.*

Proof. We will use proof by induction on n, the number of intervals.

Base case (n=1):

The base case is covered by Lemma 4, which gives us $D_f([a_1,b_1]) = (0,b_1-a_1]$. Therefore, $\mu(D_f([a_1,b_1])) = \mu([a_1,b_1]) = b_1-a_1$.

Induction Step:

Our goal is to prove that $\mu(D_f(\cup_{k=1}^{n+1}[a_k,b_k])) \geq \mu(\cup_{k=1}^{n+1}[a_k,b_k])$. Assume, without loss of generality, that $\max_{[a_1,b_1]}(|f-\lambda|) \geq \max_{[a_{n+1},b_{n+1}]}(|f-\lambda|)$. We do not lose generality because D_f is invariant with respect to horizontal reflections, i.e., $D_{f(x)} = D_{f(-x)}$. Then, by Corollary 2, $D_f([a_1,b_1] \cup [a_{n+1},b_{n+1}]) \supseteq (a_{n+1}-a_1, b_{n+1}-a_1)$. Combining this fact with Lemma 1 gives

$$\begin{aligned} D_f(\cup_{k=1}^{n+1}[a_k,b_k]) &\supseteq D_f(\cup_{k=1}^{n}[a_k,b_k]) \cup D_f([a_1,b_2] \cup [a_{n+1},b_{n+1}]) \\ &\supseteq D_f(\cup_{k=1}^{n}[a_k,b_k]) \cup (a_{n+1}-a_1, b_{n+1}-a_1). \end{aligned}$$

Next, observe that $(a_{n+1}-a_1, b_{n+1}-a_1)$ and $D_f(\cup_{k=1}^{n}[a_k,b_k])$ are disjoint. Indeed, if $d \in D_f(\cup_{k=1}^{n}[a_k,b_k])$, then $d \leq b_n - a_1 \leq a_{n+1}-a_1$. Computing the length of both sides and applying the induction hypothesis, we get

$$\begin{aligned} \mu(D_f(\cup_{k=1}^{n+1}[a_k,b_k])) &\geq \mu(D_f(\cup_{k=1}^{n}[a_k,b_k]) \cup (a_{n+1}-a_1, b_{n+1}-a_1)) \\ &= \mu(D_f(\cup_{k=1}^{n}[a_k,b_k])) + \mu((a_{n+1}-a_1, b_{n+1}-a_1)) \\ &\geq \mu(\cup_{k=1}^{n}[a_k,b_k]) + \mu((a_{n+1}-a_1, b_{n+1}-a_1)) \\ &= \mu(\cup_{k=1}^{n+1}[a_k,b_k]) \end{aligned}$$

□

Having established a lower bound on the size of D_f when f is defined on the finite union of closed intervals, we will now use a simple limiting argument to generalize to functions defined on the countable union of closed intervals.

Lemma 7. *Let $\{I_n\}$ be a countable collection of closed intervals and define $A = \cup_{n=1}^{\infty} I_n$. Assume that A is bounded and $\{I_n\}$ have disjoint interiors. Let f be a continuous function on A such that $f(x) = \lambda$ on the endpoints of each I_n and either $f(x) > \lambda$ for all $x \in A^o$ or $f(x) < \lambda$ for all $x \in A^o$. Then $\mu(D_f(A)) \geq \mu(A)$.*

Proof. Fix $\varepsilon > 0$. Since A is bounded and $\{I_n\}$ have disjoint interiors, we know that $\lim_{n\to\infty} \mu\left(\cup_{k=n}^{\infty} I_k\right) = 0$. Thus there exists some $N \in \mathbb{N}$ such that $\mu\left(\cup_{k=N}^{\infty} I_k\right) < \varepsilon$. Applying Lemma 6 and Lemma 1 yields

$$\begin{aligned}
\mu(D_f(A)) &\geq \mu\left(D_f\left(\cup_{k=1}^{N} I_k\right)\right) \\
&\geq \mu\left(\cup_{k=1}^{N} I_k\right) \\
&= \mu(A) - \mu\left(\cup_{k=N}^{\infty} I_k\right) \\
&> \mu(A) - \varepsilon
\end{aligned}$$

Therefore, $\mu(D_f(A)) \geq \mu(A)$ because ε was arbitrary. □

With access to these lemmas, we are now prepared to prove that $D_f([a,b])$ must contain at least a third of the distances in $(0, b-a]$.

Theorem 3. *If $f \in C_\lambda([a,b])$ then $\mu(D_f) \geq \frac{b-a}{3}$.*

Proof. Let $A_>$, $A_<$, and $A_=$ be the subsets of $[a,b]$ on which f is greater than, less than, and equal to λ respectively.

$A_>$ and $A_<$ are the preimages of open sets under a continuous function and are thus open. Therefore, each is a countable union of open intervals. Applying Lemma 7 to the closure of each tells us that $\mu(D_f(\overline{A_>})) \geq \mu(\overline{A_>}) = \mu(A_>)$ and $\mu(D_f(\overline{A_<})) \geq \mu(\overline{A_<}) = \mu(A_<)$[1]. Applying Lemma 2 to $A_=$ gives $\mu(D_f(A_=)) \geq \mu(A_=)$. Combining these three inequalities with Lemma 1, we have

$$\begin{aligned}
\mu(D_f([a,b])) &\geq \max\left(\mu(D_f(\overline{A_>})), \mu(D_f(\overline{A_<})), \mu(D_f(A_=))\right) \\
&\geq \max\left(\mu(A_>), \mu(A_<), \mu(A_=)\right) \\
&\geq \frac{b-a}{3}
\end{aligned}$$

where the last line follows from $\mu(A_>) + \mu(A_<) + \mu(A_=) = b - a$. □

4 Future Work

4.1 Is $\frac{b-a}{3}$ a minimum?

Theorem 3 establishes a lower bound on D_f for functions in $C_\lambda([a,b])$. The key was to restrict our attention to $A_> = \{x : f(x) > \lambda\}$ because if $f(x) = f(y)$, then either both x

[1] Dropping the closure doesn't change the length because the union of countably many intervals has a countable boundary.

and y are in $A_>$ or neither are. The same holds for $A_<$ and $A_=$. In other words, points in D_f cannot arise due to "interactions" among $A_>$, $A_<$, and $A_=$. With this in mind, the bound in Theorem 3 seems tight: simply define a function which is positive on the first third of $[a,b]$, negative on the second third, and zero on the last third. Then each of $A_>$, $A_<$, and $A_=$ should contribute $(0, \frac{b-a}{3}]$ to D_f. For example, let

$$f(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq 2\pi \\ 0 & \text{if } 2\pi \leq x \leq 3\pi \end{cases}.$$

The reason this strategy doesn't work is $A_=$. Indeed, $D_f(A_>) = D_f(A_<) = (0,\pi]$. However, $A_= = \{0,\pi\} \cup [2\pi, 3\pi]$ and $D_f(A_=) = (0,3\pi]$. This makes D_f as large as possible due to interactions between the points 0 and π and the interval $[2\pi,3\pi]$.

The trouble with the previous example is the presence of isolated points 0 and π in $A_=$. The former is unavoidable, but we can eliminate the latter by making f zero *between* the intervals on which it is positive and negative. Let

$$f(x) = \begin{cases} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } \pi \leq x \leq 2\pi \\ -\sin x & \text{if } 2\pi \leq x \leq 3\pi \end{cases}.$$

Now $A_= = \{0,3\pi\} \cup [\pi,2\pi]$ and $D_f(A_=) = (0,2\pi]$, but D_f is still strictly greater than the bound established in Theorem 3.

Had we defined D_f slightly differently to ignore the endpoints of the domain of f, the previous example would prove Theorem 3 is sharp. More precisely, if we instead define $D_f = \{d > 0 : |x-y| = d \text{ and } f(x) = f(y) \text{ for some } a < x < y < b\}$, then $D_f = (0,\pi]$ in the previous example.

However, if we stick to our original definition, is there an $f \in C_\lambda([a,b])$ with $\mu(D_f) = \frac{b-a}{3}$? If not, what is the infimum of D_f over all such f?

4.2 Generalization

What does $D_f(X)$ look like when X is not a closed interval? We could broaden the class of functions we look at by defining

$$C_\lambda(X) = \left\{ f : \overline{X} \to \mathbb{R} \mid f \text{ is continuous and } f(x) = \lambda \text{ for all } x \in \partial X \right\}.$$

What is $\bigcap D_f$ over all such f and what is the infimum of $\mu(D_f)$?

We could also explore functions with an n-dimensional domain and/or m-dimensional codomain.

What does $D_f(X)$ look like when X is n-dimensional? The more general definition of $C_\lambda(X)$ proposed above works just fine in this case. For simplicity, we might want to start with cubes or spheres, and slowly relax the constraints on X. Additionally, as in Section 4.1 we should amend the definition D_f to ignore the boundary of X. Otherwise $D_f = (0, (X)]$ always (unless X is disconnected).

What does $D_f([a,b])$ look like when the codomain of f is m-dimensional? If $m > 1$ the minimum $D_f([a,b])$ becomes $\{b-a\}$. Consider, for example, $f : [0,2\pi] \to \mathbb{R}^2$ defined by $f : x \mapsto (\cos x, \sin x)$. The only pair of points in $[0, 2\pi]$ which get mapped to the same output are 0 and 2π, so $D_f = \{2\pi\}$.

To construct an interesting generalization, we must then restrict our attention to functions mapping a closed interval to some subset $A \subset \mathbb{R}^m$. If A contains any "loops," the minimum $D_f([a,b])$ becomes $\{b-a\}$, so A should be a one-dimensional "loop-free" set.

Lastly, if the previous questions are settled, perhaps we could define

$$C_\lambda(X,Y) = \left\{ f : \overline{X} \to Y \mid f \text{ is continuous and } f(x) = \lambda \text{ for all } x \in \partial X \right\}$$

and classify D_f in terms of $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$.

Bibliography

[1] Abbott, Stephen, *Understanding Analysis,* Undergraduate Texts in Mathematics, Springer, New York, 2015.