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Viviani's Theorem, Minkowski's Theorem and Equiangular Polygons

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Abstract

Consider a polygon $P \subset \mathbb{R}^2$ and a positive real number t. The action of dilating (or shrinking) P by a factor of t is equivalent to dilating (or shrinking) each side of P by t, while preserving the unit normal vectors to the edges. A possible variation to this task is to consider elongating or shortening each side of P by t, also keeping the unit normal vectors intact. It is not clear a priori that such a task can always be accomplished. The current work addresses this adaptation and draws a connection with Viviani's theorem and equiangular polygons. The main purpose of the paper is to highlight a famous theorem of Minkowski from convex geometry that makes this connection possible and gives a generalization to higher dimensions.

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1 Introduction

Let P be a regular polygon of side length s. Then, dilating P by a factor $t > 1$ is the same as adding $(t-1)s$ to each edge, and shrinking P by a factor of $t < 1$ is the same as subtracting $(1-t)s$ from each edge. Take the square S of edge length 3 as an example. For $t = 2$, the square $2S$ has edge length $6 = 3 + (2-1)(3)$ and for $t = \frac{1}{3}$, the square $\frac{1}{3}S$ has edge length $1 = 3 - (1 - \frac{1}{3})(3)$.

It is not hard to see that the two problems are generally equivalent in the case of regular polygons. What happens when P is not regular? If P is the trapezoid with side lengths $5, 5, 5, 11$, then it is impossible to add a real number t to each of the sides while keeping the edge normal vectors intact (the reader is encouraged to try it on their own).

To this end, we aim at connecting two seemingly unrelated theorems from differents historical eras of mathematics: Viviani's theorem and Minkowski's theorem. The former dates back to the mid 17-th century; it asserts that no matter where you place a point inside a regular polygon, the sum of the distances from the point to the sides of the polygon remains constant. The latter is due to Hermann Minkowski from the early 1900's; it states that every polygon (or polytope in general) is uniquely determined, up to translation, by the directions and measures of its sides (or facets in general).

2 Viviani's Theorem

Viviani's theorem states that the sum of the distances from any interior point to the sides of an equilateral triangle is independent of the position of the point. In particular, this sum is equal to the length of the height of the triangle.

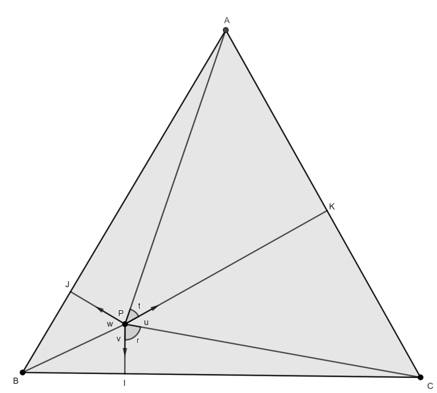

Figure 1: Equilateral Triangle ABC with an interior point P

There are many proofs and generalizations of Viviani's theorem in the literature. We survey some of them below and provide two additional elementary proofs as well. Historically, the problem of finding the Fermat point of the vertices A, B, C of a triangle ABC, i.e., the point that minimizes the sum of the distances to the vertices, was first proposed by Fermat in a private letter to Torricelli. Torricelli solved the problem and his solution was published by his student Viviani in 1659. The solution uses the fact that the sum of the distances from any point inside an equilateral triangle to its sides is constant, which is commonly known today as Viviani's theorem.

Viviani's original proof [8] (Appendix, pp. 143-150) uses areas as follows. Let ABC be an equilaterial triangle of side length s and height length h. Let P be an interior point. The area of the triangle ABC $\left(\frac{sh}{2}\right)$ is equal to the sum of the areas of the triangles ABP, BPC, and CPA. Since $AB = AC = BC = s$, then we conclude that $PJ + PI + PK = h$ (see Figure 1). In fact, Viviani proved a bit more, namely that the sum of the distances from any point inside a regular polygon to its sides is constant, and is less than the sum from any point outside the regular polygon.

Using rotations of smaller triangles inside the equilateral triangle, Kawasaki [6] proved Viviani's theorem as illustrated in Figure 2.

Chen and Liang [3] proved the converse of Viviani's theorem: if the sum of the distances from an interior point of a triangle to its sides is independent of the location of the point, then the triangle is equilateral. Moreover, they showed that the sum of the distances from an interior point to the sides of a quadrilateral is constant if and only if the quadrilateral is a parallelogram.

The area method highlighted in Viviani's original proof can be extended to show that the theorem holds for all regular polygons as well. Likewise, by a volume argument, a similar result holds for regular polyhedra in $\mathbb{R}^3$: the sum of the distances from any point inside a regular polyhedron to its faces is independent of the location of the point.

Abboud [1] defines a polygon to have the *constant Viviani sum (CVS) property* if the sum of the distances from any interior point to the sides of the polygon is constant. He then shows that a necessary and sufficient condition for a convex polygon to have such property is the existence of three non-collinear interior points with equal sums of distances. His proof relies on ideas from linear programming.

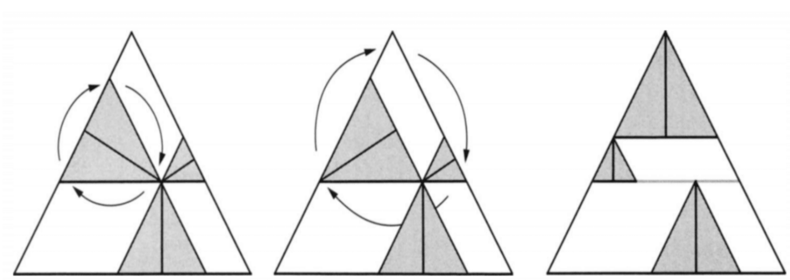

Figure 2: Kawasaki's proof using rotations [6].

We conclude this section with two new proofs of Viviani's theorem, based on simple geometric arguments.

First Proof. With the notation set above, recall that the normal vectors to the edges of the equilateral triangle ABC satisfy

$$\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w} = \overrightarrow{0}. \tag{1}$$

After taking the dot product with the vector $\overrightarrow{PA}$, we get:

$$\begin{aligned}
\overrightarrow{PA}.(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}) &= \overrightarrow{PA}.\overrightarrow{0} \\
\overrightarrow{PA}.\overrightarrow{u} + \overrightarrow{PA}.\overrightarrow{v} + \overrightarrow{PA}.\overrightarrow{w} &= 0 \\
PK + PA\Big(\cos\Big(\frac{2\pi}{3} + \angle APK\Big)\Big) + PJ &= 0 \\
PK + PA\Big(-\frac{1}{2}\cos\angle APK - \frac{\sqrt{3}}{2}\sin\angle APK\Big) + PJ &= 0 \\
PK - \frac{1}{2}PA\cos\angle APK - \frac{\sqrt{3}}{2}PA\sin\angle APK + PJ &= 0 \\
PK - \frac{1}{2}PK - \frac{\sqrt{3}}{2}AK + PJ &= 0.
\end{aligned}$$

This leads to the following result

$$\frac{1}{2}PK + PJ - \frac{\sqrt{3}}{2}AK = 0. \tag{2}$$

Similarly, multiplying Equation (1) by the vector $\overrightarrow{PC}$, we obtain

$$\frac{1}{2}PK + PI - \frac{\sqrt{3}}{2}CK = 0. \tag{3}$$

Finally, adding Equations (2) and (3), we get

$$\begin{aligned}
\frac{1}{2}PK + PJ - \frac{\sqrt{3}}{2}AK + \frac{1}{2}PK + PI - \frac{\sqrt{3}}{2}CK &= 0 \\
PK + PJ + PI &= \frac{\sqrt{3}}{2}(AK + CK) \\
PK + PJ + PI &= \frac{\sqrt{3}}{2}AC \\
PK + PJ + PI &= h.
\end{aligned}$$

□

Next we embed Figure 1 in cartesian coordinates and provide yet another proof of Viviani's theorem.

Second Proof. Without loss of generality, we may assume that $B(0,0)$ and $C(s,0)$ for some positive real number s. Since ABC is an equilateral triangle, the point A has coordinates $\left(\frac{s}{2}, \frac{s\sqrt{3}}{2}\right)$ and the line segments BC, AB, AC have equations $y = 0$, $\sqrt{3}x - y = 0$, $\sqrt{3}x + y - \sqrt{3}s = 0$, respectively. Let $P(x,y)$ be a point inside the triangle ABC. Using the formula for the distance from a point to a line, we get $PI = y$, $PK = \frac{-\sqrt{3}x - y + \sqrt{3}s}{2}$, and $PJ = \frac{\sqrt{3}x - y}{2}$. Adding the three lengths together leads to $PK + PJ + PI = \frac{\sqrt{3}}{2}s = h$. □

As a side note, we mention a remarkable application of Viviani's theorem in chemistry.

Consider a mixture of three chemical components represented by the vertices of an equilateral triangle. If the height of the triangle is taken as unity and the mixture is depicted by a point inside the triangle, then the distances from this point to the sides correspond to the proportions of the components in the mixture. The same principles can be applied to a system of four components: within a regular tetrahedron whose vertices represent the pure components, the distances from an interior point to the faces again sum to a constant, and may be used to represent the proportions. For further details, the reader is referred to the book [4] (Chapter 8).

3 Minkowski's Theorem

Polytopes are the generalization of polygons in higher dimensions. Formally, a convex polytope is the convex hull of a finite set of points in $\mathbb{R}^n$, or equivalently, the intersection of a finite number of hyperplanes.

The Minkowski problem for polytopes concerns the following specific question. Given a collection $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$ of unit vectors and $a_1, \ldots, a_k > 0$, under what condition(s) does there exist a polytope P having the $\overrightarrow{u_i}$'s as its facet normal vectors and the a_i's as its facet areas? The answer to this question is known as the Minkowski's existence and uniqueness theorem for polytopes.

Theorem 1 (Minkowski)**.** *Let $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$ be unit vectors that span $\mathbb{R}^n$, and $a_1, \ldots, a_k >$ 0. Then there exists a polytope P in $\mathbb{R}^n$ having facet unit normal vectors $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$ and corresponding facet areas $a_1, \ldots, a_k$ if and only if*

$$a_1\overrightarrow{u_1} + \cdots + a_k\overrightarrow{u_k} = \overrightarrow{0}. \tag{4}$$

Moreover, this polytope is unique up to translation.

Minkowski's original proof involves two steps. First, the existence of a polytope satisfying the given facet data is demonstrated by a linear optimization argument. In the second step, the uniqueness of that polytope (up to translation) is shown by a generalized isoperimetric inequality for mixed volumes. Alternative proofs, generalizations, and applications of Minkowski's theorem are abundant in the literature. We refer the reader to [7] and the references therein for a good exposition on this topic.

Note that in the 2-dimensional Euclidean space, the facet areas of a polygon are simply the edge lengths of the polygon. In the case of equilateral triangles, it is clear that Equation (4) is equivalent to Equation (1).

A special family of polygons are the *equiangular* polygons. These are characterized by having equal angles without necessarily having congruent edges. For a set of positive real numbers $a_1, \ldots, a_k$, it is well known [2] that there exists an equiangular polygon with side lengths $a_1, \ldots, a_k$ if and only if the polynomial $a_1 + a_2x + \cdots + a_kx^{k-1}$ vanishes at $e^{\frac{2\pi}{k}i}$. Hence, for example, equilateral triangles are the only equiangular triangles and rectangles are the only equiangular quadrilaterals.

We prove this result using Minkowski's theorem as follows. Let P be a polygon in $\mathbb{R}^2$ with side lengths $a_1, \ldots, a_k$ and interior angle measures $\theta_1, \ldots, \theta_k$ ($k \geq 3$). Recall that $\theta_1 + \cdots + \theta_k = (k-2)\pi$ for any k-gon. Consider the following polynomial in $k-1$

variables

$$\begin{aligned} p(x_1, x_2, \ldots, x_{k-1}) &:= a_1 + a_2 x_1 + \cdots + a_k x_1 \ldots x_{k-1} \\ &= a_1 + \sum_{i=1}^{k-1} a_{i+1} x_1 \ldots x_i. \end{aligned}$$

With the definition above, Minkowski's theorem in dimension 2 can be written in algebraic form. Observe that the angle formed by the vectors $\overrightarrow{u_j}$ and $\overrightarrow{u_{j+1}}$ is equal to $\pi - \theta_j$, for each j. Since each $\overrightarrow{u_j}$ is a unit vector, we can then write the vector $\overrightarrow{u_j} = e^{i[(\pi-\theta_1)+(\pi-\theta_2)+\cdots+(\pi-\theta_{j-1})]}$ for $j = 2, \ldots, k$ (we consider $\overrightarrow{u_1}$ the vector of reference here). By substituting the latter expression of $\overrightarrow{u_j}$ in Equation (4), we get the following theorem.

Theorem 2. *Let $a_1, \ldots, a_k$ and $\theta_1, \ldots, \theta_k$ be positive real numbers such that $\theta_1 + \cdots + \theta_k = (k-2)\pi$. Then, there exists a polygon with edge lengths $a_1, \ldots, a_k$ and interior angle measures $\theta_1, \ldots, \theta_k$ if and only if the polynomial $p(x_1, x_2, \ldots, x_{k-1})$ vanishes at $(e^{i(\pi-\theta_1)}, e^{i(\pi-\theta_2)}, \ldots, e^{i(\pi-\theta_{k-1})})$.*

If P is equiangular, then $\theta_1 = \cdots = \theta_k = \frac{k-2}{k}\pi$. This implies that $\pi - \theta_i = \frac{2\pi}{k}$ for $i = 1, \ldots, k$. The following can then be deduced.

Corollary 3. *There exists an equiangular polygon with edge lengths $a_1, \ldots, a_k > 0$ if and only if $a_1 + a_2 e^{\frac{2\pi}{k} i} + a_3 e^{\frac{4\pi}{k} i} + \cdots + a_k e^{\frac{2(k-1)\pi}{k} i} = 0$.*

4 Viviani Polytopes

Similar to the CVS property defined above, Zhou [9] introduced the notion of Viviani polytopes as follows. Let $p_1, \ldots, p_k$ be distinct hyperplanes enclosing a convex polytope $P \subset \mathbb{R}^n$, and $\overrightarrow{u_1}, \ldots, \overrightarrow{u_k}$ the outward unit normal vectors to each p_i, respectively. For a point $T \in \mathbb{R}^n$, denote by d_i the signed distance from T to the hyperplane p_i and let $v(P) := \sum_{i=1}^{k} d_i$. We call P a *Viviani polytope* if v is a constant function, i.e. independent of the choice of the point T.

The main result in [9] is a geometric characterization of Viviani polytopes in any dimension. An algebraic characterization using linear programming was previously derived in [1].

Theorem 4 (Theorem 1 in [9])**.** *With the above notation, a polytope $P \subset \mathbb{R}^n$ is Viviani if and only if*

$$\overrightarrow{u_1} + \cdots + \overrightarrow{u_k} = \overrightarrow{0}. \tag{5}$$

In light of Theorem 2, a polynomial formulation for Viviani polygons can be derived as follows. Given a set of positive real numbers $\theta_1, \ldots, \theta_k$ that add up to $(k-2)\pi$, there exists a polygon with interior angle measures $\theta_1, \ldots, \theta_k$ if and only if $(e^{i(\pi-\theta_1)}, e^{i(\pi-\theta_2)}, \ldots, e^{i(\pi-\theta_{k-1})})$ is a root of the polynomial $1 + x_1 + x_1 x_2 + \cdots + x_1 x_2 \ldots x_{k-1}$. In particular, we get the following corollary.

Corollary 5. *Equilateral triangles are the only Viviani triangles and parallelograms are the only Viviani quadrilaterals. Moreover, equiangular polygons are Viviani for any number of sides.*

As mentioned in the first section, it can be shown that regular polygons in $\mathbb{R}^2$ and regular polyhedra in $\mathbb{R}^3$ are Viviani using an area and a volume argument, respectively. Along the same line of thought, it was shown in [5] that any polyhedron with faces of equal area is Viviani. We extend this result to all dimensions using Minkowski's theorem.

Consider a polytope $P \subset \mathbb{R}^n$ with facet unit normal vectors $\overrightarrow{u_1}, \dots, \overrightarrow{u_k}$. If the facets of P have equal area (i.e. $(n-1)$-dimensional volume), then $a_1 = \dots = a_k$ in the statement of Theorem 1, which implies that $\overrightarrow{u_1} + \dots + \overrightarrow{u_k} = \overrightarrow{0}$. By Theorem 3, one can deduce that the polytope P is Viviani. Thus, we proved the following general result.

Theorem 6. *Any polytope whose facets have equal area is Viviani.*

Finally, we turn back to our original question. Assume $P \subset \mathbb{R}^2$ is a polygon with side lengths $s_1, \dots, s_k$ and unit normal vectors $\overrightarrow{u_1}, \dots, \overrightarrow{u_k}$. The goal is to find another polygon P' with the same unit normal vectors but with side legths $s_1 \pm t, \dots, s_k \pm t$. Applying Minkowski's theorem to P and P', we get $s_1\overrightarrow{u_1} + \dots + s_k\overrightarrow{u_k} = \overrightarrow{0}$ and $(s_1 \pm t)\overrightarrow{u_1} + \dots + (s_k \pm t)\overrightarrow{u_k} = \overrightarrow{0}$, respectively. Combining the two equations, we obtain $\pm t(\overrightarrow{u_1} + \dots + \overrightarrow{u_k}) = \overrightarrow{0}$. This is equivalent to P (or P') being Viviani!

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