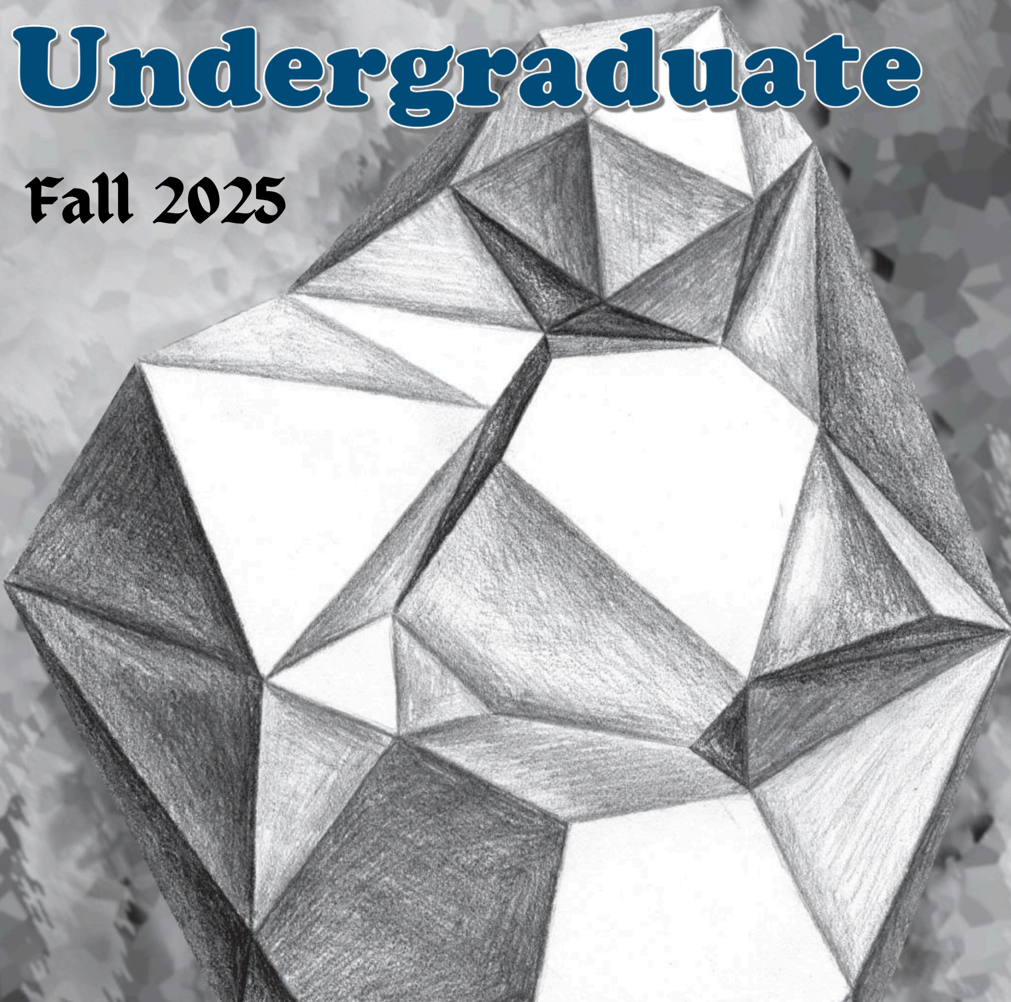
Ball State Undergraduate Mathematics Exchange



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Cover design by Patrick Foley.

A Word from the Editor

The editorial board of the *Mathematics Exchange* is pleased to present this year's issue, featuring three engaging articles that showcase the breadth and depth of contemporary undergraduate mathematical research. We extend our sincere appreciation to the authors for their dedication and creativity, and we hope that readers will find this collection both inspiring and illuminating.

Our opening article addresses the timely and impactful topic of waterborne disease transmission. The authors develop a deterministic compartment model that incorporates both human-to-human and water-to-human interactions through a river with distinct upstream and downstream access points. By examining intervention strategies through sensitivity analyses and exploring their effects on both epidemic and endemic phases, the paper provides valuable insights for improving public health responses and prioritizing control measures.

The second article ventures into the elegant world of graph labelings, focusing on odd graceful labelings of prism graphs $C_n \times P_2$ for even n. Building on established cases, the authors construct explicit labeling schemes for $n = 6k$, $n = 6k+2$ and $n = 6k+4$, offering clear methods and patterns that contribute to the broader study of graceful and odd graceful labelings.

The final article examines a generalization of the well-known bondage number of a graph by introducing the $k-$ synchronous bondage number. After an in-depth discussion of the $2-$ synchronous case, the authors extend their analysis to general k, providing values for several graph families and establishing bounds for general graphs. Their work highlights the potential of this parameter as a meaningful measure of connectivity with applications to network design and optimization.

We hope that this issue of the *Mathematics Exchange* offers you fresh perspectives and renewed enthusiasm for mathematical inquiry. As always, we warmly welcome your feedback and suggestions as we continue striving to enrich the experiences of our readers.

Yayuan Xiao

11.17.2025

Call for Papers

We are always soliciting contributions for future issues of this journal. Contributions are accepted from all undergraduate students who have worked on a project beyond the classroom in any mathematical area (e.g., pure, applied, actuarial, and education). Appropriate papers from other departments and other institutions are also welcome. Often the articles are written by undergraduates individually, working in teams, or working with faculty. On occasion we also include articles written solely by faculty or graduate students as long as they are accessible to undergraduates.

To submit an article, please select ONE member from the editorial board, and forward your material in PDF form, usually prepared by LaTeX (preferred) or Microsoft Word, to the editor you selected. We use double anonymized peer review, the identities of both reviewers and authors are concealed from each other throughout the review. To facilitate this, please remove any identifying information, such as authors' names or affiliations, from your manuscript before submission. Please ensure that the title page (that include all authors' names and affiliations, a complete address of the corresponding author including an email address, acknowledgements, and conflict of interest statement) is present in your submission as a separate file. If authors are undergraduate students, please include your advisor's name and contact information in the title page. Review and selection of articles is handled by the editorial committee. Editorial changes of accepted articles are communicated through students' advisors, when appropriate.

More information, including links to all previous issues, are available online at
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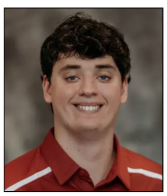
Waterborne Disease Dynamics with a River

*Jackson Leeper, Chad Westphal**



Jackson Leeper worked on this paper as part of a summer research internship between his Sophomore and Junior years at Wabash College. He graduated in 2025 with a double major in Computational Mathematics and Computer Science and hopes to use the analytical tools and the thought process in this research to conduct research in a career of data science.

Chad Westphal is a Professor of Mathematics and Computer Science at Wabash College. His research is primarily in numerical analysis and has enjoyed working with smart and ambitious students in various projects at the intersection of deterministic and stochastic modeling, making good use of tools from dynamical systems, probability, and network analysis.

Abstract

A wide range of waterborne diseases spread through a population through both human-to-human interaction and water-to-human interaction. In this paper, we propose a deterministic compartment model to simulate the transmission of a waterborne pathogen through a population whose common water source is a river with both upstream and downstream access points. This allows for a distinction between drinking and shedding behavior with respect to the upstream and downstream water sources. We consider the effectiveness of several intervention methods with respect to two metrics: the basic reproductive number, R_0, in the epidemic phase and the steady-state infected population fraction, i_∞, in the endemic phase. Using both local and global sensitivity analysis techniques, the relative effectiveness of interventions are demonstrated, leading to a clearer understanding of how to prioritize efforts to either prevent an epidemic or to reduce the endemic level of disease in a population.
Keywords: Compartmental Model, Waterborne Disease, Cholera, Basic Reproductive Number, Parameter Sensitivity
AMS Subject Classification: 92D30, 34C60

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1 Introduction

Waterborne diseases are a present day threat to the health of many people worldwide. Around 2 billion people rely on water sources that may be contaminated with fecal matter, and many diseases, including cholera, dysentery, polio, and others, cause an estimated $485,000$ diarrheal deaths each year [13]. In the U.S., around 7.2 million people get sick from waterborne pathogens per year [6]. It is thus important to understand the dynamics of waterborne diseases and the effectiveness of intervention methods for populations with shared water sources that may become infected.

We consider a compartment model similar to work in [7, 10, 12, 14, 16, 15]. These authors mainly use a SIWR model, which is a common framework for waterborne diseases. The SIWR approach typically includes a single point source for water. In [7], Fung discusses the idea of a river with a flow but his model for cholera transmission only has people getting infected from one source of water. The model we propose in this paper represents a more general approach for waterborne diseases with a flowing water source. Our model includes coupled upstream and downstream water categories. This allows our system to model two scenarios: (1) a river with both upstream and downstream access points, or (2) a recirculating water source with a filter.

We include a wide range of possible interventions through a set of tuneable parameters to effectively study the relative merits of policies designed to prevent or eliminate the transmission of the disease. Similar to [12], we include both symptomatic and asymptomatic infected categories into our model to allow for a distinction in human behavior between those aware or unaware of their ability to spread the disease. Detailed analysis on interventions for related models can be found in [2, 1, 12, 11].

The organization of this paper is as follows. In section section 2 we give a detailed overview of our mathematical model and the relevant assumptions of the disease transmission. We define a set of model parameters and their assumed ranges, and provide an initial solution of the model for a set of default baseline parameters. In section section 3 we use analytical methods to describe key features in an outbreak scenario. We use the next-generation matrix approach [4] to derive the basic reproductive number, R_0, providing insight into the beginning of an outbreak. We also investigate equilibrium solutions of the model to define the steady state level of infection in the endemic state, i_∞ [9]. In section section 4 we perform parameter sensitivity analysis to understand how to effectively decrease R_0 and i_∞ through combinations of interventions. In section section 5 we close with a discussion of the main results and conclusions from both mathematical modeling and public policy points of view.

2 Mathematical Model

We develop our model using ideas similar to those in [7, 10, 12, 14, 16, 15], which focus on aspects of waterborne disease dynamics. We note that, while this model is not limited to a specific disease, cholera transmission is largely consistent with the main ideas we pursue here. See [14, 12] for examples of cholera models. The the key feature of these models are the ability of the disease to spread through both human-to-human and human-to-water interactions. Figure 1 below illustrates our compartment model.

We assume that people infected with the disease may be symptomatic (denoted I_s) or asymptomatic (denoted I_a) and that people show symptoms with probability

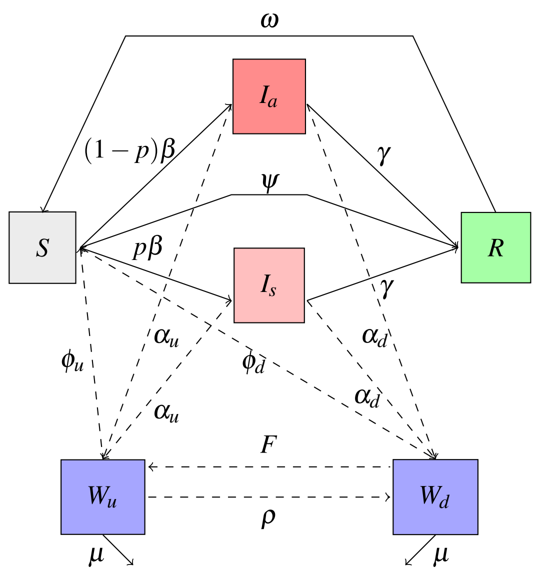


Figure 1: Compartment model illustration for system (1).

p. A person susceptible to the disease (denoted S) may become infected through an interaction with someone from either infected category with contact rate parameters β_s and β_a. This mode of transmission is typically through bodily fluids. A susceptible person may also become infected by interacting with water from either the upstream or downstream source with contact rate parameters ϕ_u and ϕ_d. The likelihood of infection from ingesting water depends on the relative pathogen concentration (denoted W_d and W_u). We assume a natural death rate of the water pollutant of μ. We also assume a uniform flow from upstream to downstream that transports water into and out of each water source with rate ρ. Once a person becomes infected, they shed waste into each water source with rate parameters α_u and α_d. If we assume $\alpha_u = \alpha_d$ and $\phi_u = \phi_d$, then interactions with the two water sources are symmetric and only the flow provides a distinction between upstream and downstream.

We typically assume that our water source is similar to a flowing river, where the flow is simply from upstream to downstream. We also include the possibility of a different configuration that models a water source with a recirculation filter. Under this assumption, water from the downstream source is filtered and returned to the upstream source. The effectiveness of filtration is denoted with the parameter F, where $F = 0$ represents a perfect filter (i.e., uncontaminated water is returned upstream), $F \in (0,1)$ represents an imperfect filter, and $F = 1$ represents a return of unfiltered water. Thus, $F = 0$ also corresponds to the default river configuration with no possibility of pollutant recirculation.

After being infected for some period of time, determined by parameter γ, the individual becomes recovered (denoted R). While in this category, an individual is immune to the disease. However, this immunity may not be permanent and how long the immunity lasts is determined by the parameter ω. Once the immunity is gone, the individual becomes susceptible again. We also include the possibility of intervention through vaccination, where the susceptible population is vaccinated at a rate determined by the parameter ψ. Under vaccination, susceptible individuals are moved directly to the recovered/removed compartment, where they remain immune for the same time period as one newly recovered from an infection would.

We denote the total number of people in the model by $N = S + I_s + I_a + R$, which remains constant over time. The water compartments behave differently since they are not a part of the population. Both W_u and W_d are calibrated to represent a relative concentration scaled with $W_u, W_d \in [0,1]$. A value of 0 represents uncontaminated water, and 1 corresponds to a maximum concentration equivalent to pure wastewater. Under this scaling, the parameters α_u and α_d can be interpreted as the increase in pollutant concentration from one infected person in one day, when shedding into initially uncontaminated water. Shedding into maximally contaminated water does not increase the concentration. Likewise, the parameters ϕ_u and ϕ_d can be interpreted as the probability of infection for one susceptible individual who ingests the average daily amount of water that is maximally contaminated. It should be noted that these parameters reflect an overall assumption on the size of the water source as well as an average rate of interaction between people and water.

The ODE system for our model is given by

$$
\begin{aligned}
\frac{dS}{dt} &= -(\beta_s I_s + \beta_a I_a)S/N - \phi_u S W_u - \phi_d S W_d + \omega R - \psi S \\
\frac{dI_a}{dt} &= (1-p)((\beta_s I_s + \beta_a I_a)S/N + (\phi_u W_u + \phi_d W_d)S) - \gamma I_a \\
\frac{dI_s}{dt} &= p\,((\beta_s I_s + \beta_a I_a)\,S/N + (\phi_u W_u + \phi_d W_d)S) - \gamma I_s \\
\frac{dR}{dt} &= \gamma(I_a + I_s) - \omega R + \psi S \\
\frac{dW_u}{dt} &= \alpha_u(1 - W_u)(I_a + I_s)/N - \mu W_u - \rho W_u + F\rho W_d \\
\frac{dW_d}{dt} &= \alpha_d(1 - W_d)(I_a + I_s)/N - \mu W_d + \rho W_u - \rho W_d,
\end{aligned}
\tag{1}
$$

which is well-posed when closed by a set of nonnegative initial conditions consistent with $N = S + I_s + I_a + R$ and $W_u, W_d \in [0,1]$ and nonnegative parameters. Over time, human populations will each remain nonnegative with constant N. Water concentrations each remain bounded in $[0,1]$.

Table 1 summarizes the parameters in our model, the permissible range of each, and the default values and units that we use for numerical simulations.

We note that while this model is not specifically calibrated by data on a specific disease, it shares the basic structure and transmission dynamics with cholera, typhoid, and polio. Some of the parameters used in this paper are chosen empirically since they represent a combination of human behavior (e.g., drinking and shedding rates α_u, α_d, ϕ_u, ϕ_d, and vaccination rate ψ) and geographically specific conditions (e.g., river flow rate ρ and filter parameter F). Other parameters are more directly aligned with the particular disease (e.g., transmission rates β_a, β_s, recovery rate γ, pathogen death rate μ, symptomatic probability p, and waning immunity rate ω). In these cases our

Parameter	Range	Default Value
β_a	$[0,\infty)$	0.25 day^{-1}
β_s	$[0,\infty)$	0.2 day^{-1}
γ	$[0,\infty)$	0.1 day^{-1}
α_u	$[0,\infty)$	0.15 day^{-1}
α_d	$[0,\infty)$	0.15 day^{-1}
ϕ_u	$[0,\infty)$	0.2 day^{-1}
ϕ_d	$[0,\infty)$	0.2 day^{-1}
ω	$(0,\infty)$	0.02 day^{-1}
ρ	$[0,\infty)$	0.05 day^{-1}
p	$[0,1]$	0.4
μ	$[0,\infty)$	0.08 day^{-1}
ψ	$[0,\infty)$	0 or 0.01 day^{-1}
F	$[0,1]$	0

Table 1: Ranges, default values, and units of parameters in the model.

parameter selection and associated control ranges were done to be consistent with work in the related literature [7, 15, 16, 12, 2, 8].

Figure 2 shows the solution to system (1) with default parameters and initial conditions $S = 100, I_a = I_s = R = 0, W_u = 0.5, W_d = 0$. We see that this scenario results in an active epidemic phase lasting approximately $40-60$ days, leading into an endemic phase with a consistent portion of the population infected. We also note that each water source remains contaminated in the endemic phase, with a slightly higher asymptotic pollutant level downstream.

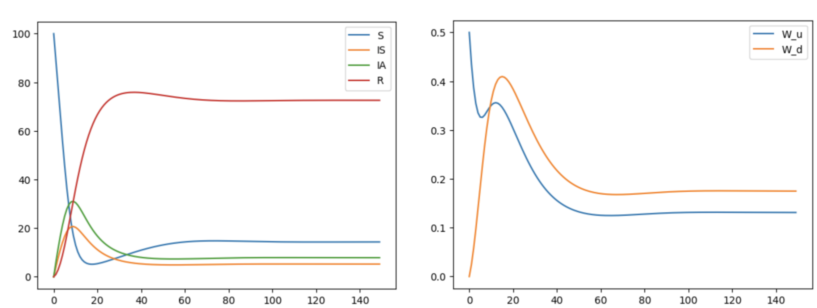


Figure 2: Solution of system (1), human populations (left) and water concentrations (right), for 150 days after an initial water contamination is introduced.

3 Analytical Results

In this section we consider equilibrium solutions and the basic reproductive number for our model. Understanding these concepts analytically provides a basis for predicting the dynamics of an outbreak scenario and gives insights toward potential intervention efforts.

System (1) has two physically relevant equilibrium solutions for parameters in the permissible range. The disease-free equilibrium represents the state with no pathogen present. The second equilibrium solution corresponds to the endemic phase of the disease outbreak. This solution depends on parameter values and is algebraically cumbersome in its general form. Using a value of $N = 100$ and default parameter values from Table 1 results in the equilibrium solution values in Table 2 below. Note that the endemic equilibrium values correspond to the horizontal asymptotes of Figure 2.

	S	I_a	I_s	R	W_u	W_d
Disease-Free Equilibrium	100	0	0	0	0	0
Endemic Equilibrium	12.09	8.19	5.46	74.26	0.13	0.17

Table 2: Values of the two equilibrium solutions with default parameters and $N = 100$.

The stability of the equilibrium solutions can help predict how the system will evolve over time. When parameters are in the permissible range, the endemic solution will be stable. The stability of the disease-free equilibrium solution depends on parameter values and is the basis for deriving the basic reproductive number.

The basic reproduction number, R_0, is the epidemiological concept of how many new infections are generated from each infected individual in the early stages of a potential outbreak. We use the next-generation matrix approach to derive an analytical determination of R_0. A detailed description of this method can be found in [4].

We first compute the Jacobian of the right-hand side of system 1, evaluated at the disease-free equilibrium, given by

$$J = \begin{pmatrix} -\psi & -\beta_a & -\beta_s & \omega & -\phi_u N & -\phi_d N \\ 0 & (1-p)\beta_a - \gamma & (1-p)\beta_s & 0 & (1-p)\phi_u N & (1-p)\phi_d N \\ 0 & p\beta_a & p\beta_s - \gamma & 0 & p\phi_u N & p\phi_d N \\ \psi & \gamma & \gamma & -\omega & 0 & 0 \\ 0 & \alpha_u/N & \alpha_u/N & 0 & -\mu-\rho & F\rho \\ 0 & \alpha_d/N & \alpha_d/N & 0 & \rho & -\mu-\rho \end{pmatrix}. \quad (2)$$

Now consider the rows and columns of that correspond to the infectious components of the system, I_a, I_s, W_u, W_d, and define the resulting 4×4 matrix as $\tilde{J}$. Splitting this into transmission (T) and transition (Σ) components gives $\tilde{J} = T + \Sigma$ with

$$T = \begin{pmatrix} (1-p)\beta_a & (1-p)\beta_s & (1-p)\phi_u N & (1-p)\phi_d N \\ p\beta_a & p\beta_s & p\phi_u N & p\phi_d N \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

and

$$\Sigma = \begin{pmatrix} -\gamma & 0 & 0 & 0 \\ 0 & -\gamma & 0 & 0 \\ \alpha_u/N & \alpha_u/N & -\mu-\rho & F\rho \\ \alpha_d/N & \alpha_d/N & \rho & -\mu-\rho \end{pmatrix}. \tag{4}$$

These matrices generally represent the rates of infection into and out of the overall system. The next-generation matrix $K = -T\Sigma^{-1}$ thus represents a sense of the ratio of these rates. The basic reproductive ratio is given by the spectral radius of K and provides an intuitive epidemic threshold. A value of $R_0 > 1$ indicates a phase of exponential growth in the infected components while $R_0 < 1$ indicates exponential decay in the infected components. For our system we may express the overall reproductive ratio by

$$R_0 = \frac{(1-p)\beta_a + p\beta_s}{\gamma} + \frac{(\mu+\rho)(\alpha_u\phi_u + \alpha_d\phi_d) + \rho(\alpha_u\phi_d + F\alpha_d\phi_u)}{\gamma(\mu^2 + 2\mu\rho + (1-F)\rho^2)}. \tag{5}$$

The first term in R_0 accounts for human-human transmission and the second term accounts for human-water transmission. Using the default values for parameters described in Table 1 results in a default value of

$$R_0 = 7.8,$$

which combines a human-human and human-water components of 2.3 and 5.5, respectively. Further, we note that using $F = 1$ increases the value to $R_0 = 9.8$.

To compare with specific infectious diseases, we note that the basic reproductive number for cholera is 1.7-2.6, polio is 5˘7, measles is 12˘14, and pertussis is 12–17 [5, 3]. The basic reproductive number is a metric that represents conditions present at the beginning of a potential outbreak scenario, and incorporates biological, behavioral, and environmental factors. As such, specific diseases may have different effective R_0 values in different situations. For example, [3] highlights this variation as well as the challenge in estimating the basic reproductive number from measurable epidemiological data.

The analytical formula for R_0 provides a way to understand how the system parameters affect the dynamics of an outbreak. In the next section, we explore this in detail.

4 Parameter Sensitivity and Interventions

A primary motivation in the development of a mathematical model for a disease is to be able to design meaningful intervention strategies. To this end, we focus on two metrics that can help determine the effectiveness of interventions. The first metric is the basic reproductive number, R_0. This quantity provides insight to the beginning of a potential outbreak, addressing the question, “will an epidemic occur in the near future?" The second metric we consider is defined by

$$i_\infty = \lim_{t\to\infty} (I_a + I_s)/N, \tag{6}$$

which provides insight to the possibility of an endemic phase, addressing the question, “will the disease persist in the population over a long time period?" We consider 7 intervention strategies with the goal of reducing R_0 below a value of 1 and to minimize i_∞. As described in section 3, with no interventions, using the default parameters leads to values of $R_0 = 7.8$ and $i_\infty = 0.137$. Each intervention is treated as a unitless constant that modifies a parameter in the original ODE system (1). Each of the interventions and the range of values considered are summarized in Table 3 below.

Constant	Range	Purpose	Modified Parameter
c_1	$[0,1]$	Hygiene	$c_1\beta_s \to [0.0,0.2]$
c_2	$[1,3]$	Medical Treatment	$c_2\gamma \to [0.1,0.6]$
c_3	$[1,3]$	Water Treatment	$c_3\mu \to [0.08,0.24]$
c_4	$[0,2]$	Shedding Behavior	$c_4\alpha_u, c_4\alpha_d \to [0.0,0.3]$
c_5	$[0,2]$	Drinking Behavior	$c_5\phi_u, c_5\phi_d \to [0.0,0.4]$
c_6	$[0,2]$	Vaccination	$c_6\psi \to [0.0,0.02]$
c_7	$[0,2]$	Flow	$c_7\rho \to [0.0,0.1]$

Table 3: Purpose and Default values of Interventions

The resulting modified ODE system is given by

$$\begin{aligned}
\frac{dS}{dt} &= -(c_1\beta_s I_s + \beta_a I_a)S/N - (2-c_5)\phi_u S W_u - c_5\phi_d S W_d + \omega R - c_6\psi S \\
\frac{dI_a}{dt} &= (1-p)((c_1\beta_s I_s + \beta_a I_a)S/N + (2-c_5)\phi_u S W_u + c_5\phi_d S W_d) - \gamma I_a \\
\frac{dI_s}{dt} &= p((c_1\beta_s I_s + \beta_a I_a)S/N + (2-c_5)\phi_u S W_u + c_5\phi_d S W_d) - c_2\gamma I_s \\
\frac{dR}{dt} &= \gamma(I_a + c_2 I_s) - \omega R + c_6\psi S \\
\frac{dW_u}{dt} &= (2-c_4)\alpha_u(1-W_u)(I_a+I_s)/N - c_3\mu W_u - c_7\rho W_u + F c_7\rho W_d \\
\frac{dW_d}{dt} &= c_4\alpha_d(1-W_d)(I_a+I_s)/N - c_3\mu W_d + c_7\rho W_u - c_7\rho W_d.
\end{aligned} \tag{7}$$

Assumptions for each of the interventions are given in the following:

- The parameter $c_1 \in [0,1]$ reduces the contact rate between the susceptible and symptomatic infected populations. Public health efforts can educate people to be aware of the danger in contact with people who are showing symptoms of the disease. The number range of this parameter would decrease the β_s value to simulate this intervention. We assume no change in the contact rate with the asymptomatic population.

- The parameter $c_2 \in [1,3]$ represents efforts to administer medical treatment to people showing symptoms of the disease, which will decrease the average recovery time for these individuals. The number range of this parameter would increase the γ value to simulate this intervention. We assume no change in the recovery rate for the asymptomatic population.

- The parameter $c_3 \in [1,3]$ increases the death rate of the pathogen in each water source which reflects efforts to apply a water treatment routine. The number range of this parameter would increase the μ value to simulate this intervention.

- The parameters $c_4, c_5 \in [0,2]$ represent modifications in how people utilize the water sources for drinking and shedding. At the default values of $c_4 = c_5 = 1$, people will access the upstream and downstream water sources for both drinking and shedding with equal rates. Choosing either value in the range $[0,2]$ represents a proportional shift in usage from purely upstream (0) to purely downstream (1). For example, $c_4 = 1.5$ and $c_5 = 0.2$ indicates that the population is primarily shedding in the downstream water source and drinking from the upstream source.

- The parameter $c_6 \in [0,2]$ represents an effort to administer a vaccination routine to the population. When a vaccine is considered, we assume a default deployment rate of $\psi = 0.01 day^{-1}$. Choosing $c_6 = 0$ is consistent with no vaccination and $c_6 = 2$ is consistent with a vaccination campaign that innoculates 2% of the susceptible population per day, starting at the beginning of an outbreak.

- The parameter $c_7 \in [0,2]$ represents a modification of the flow rate of the water from upstream to downstream. The number range of this parameter would increase or decrease the μ value to simulate this intervention.

We first consider the implementation of each intervention in isolation. Figure 3 shows how each intervention affects the value of R_0, where the other parameters are held constant at default values.

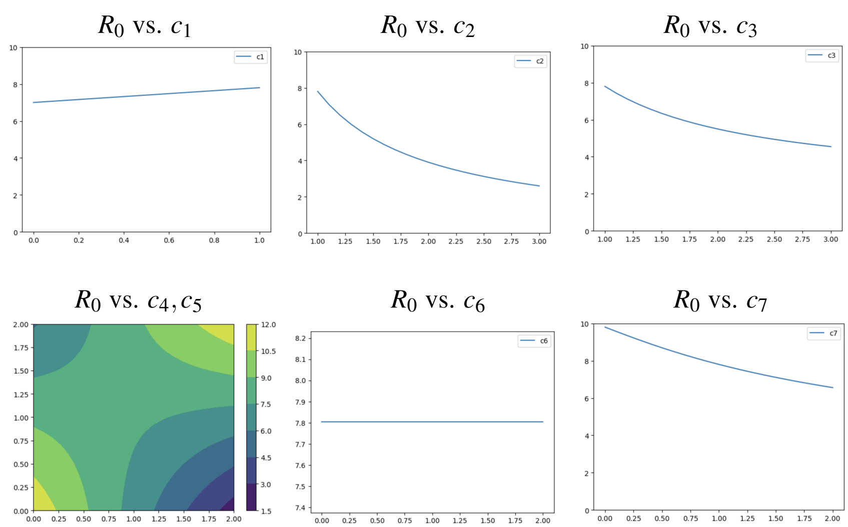


Figure 3: Control parameter local sensitivity for R_0.

It is clear that in most cases, the reduction on R_0 is marginal, with two notable exceptions. Aggressively speeding up the recovery time for symptomatic individuals with $c_2 = 3$ can reduce the basic reproductive number to $R_0 = 2.6$. The contour plot in Figure 3 shows the dependence of R_0 on c_4 (horizontal axis) and c_5 (vertical axis) simultaneously. The coupled effects of shedding and drinking behavior reveal intuitive results. Points $(0,0)$ and $(2,2)$ correspond to using one water source exclusively for both shedding and drinking, resulting in an increase in the basic reproductive number to $R_0 = 11.5$. At $(0,2)$, we have a value of $R_0 = 5.85$, where people are exclusively shedding upstream and drinking downstream. But the most advisable behavior is clearly at $(2,0)$, which results in $R_0 = 2.3$, and represents exclusively drinking from the upstream water source and shedding into the downstream source.

Next, we test each intervention in isolation for i_∞, as shown in Figure 4.

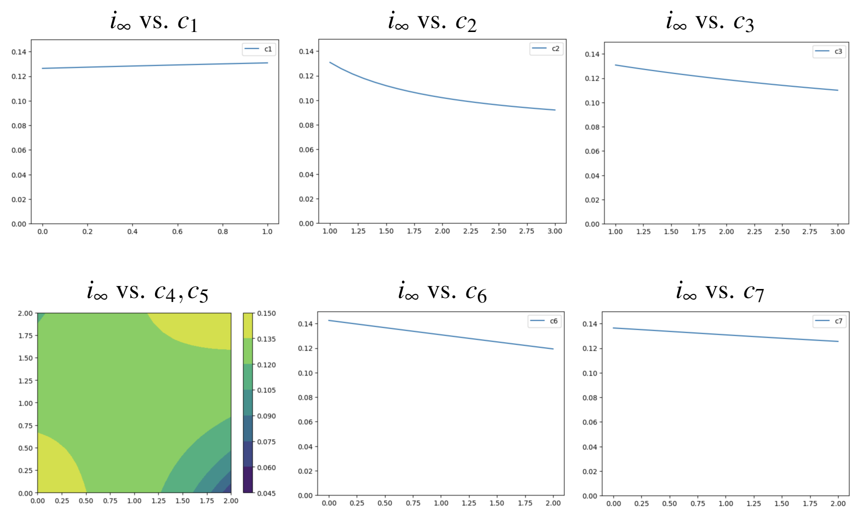


Figure 4: Control parameter local sensitivity for i_∞.

We again see that most interventions have only marginal benefit in reducing i_∞ in isolation. With similar reasoning to the sensitivity to R_0, choosing $(c_4, c_5) = (2,0)$ results in a reduction to $i_\infty = 0.05$.

One additional result of note is that c_6 affects i_∞, even though it does not affect R_0. Thus, implementing a vaccination routine at the onset of an outbreak isn't likely to prevent an epidemic, but it can be effective in reducing the persistence of the disease in the endemic period.

Now, rather than analyzing a rate of vaccine administration, we consider the question of a herd immunity threshold – what fraction of the total population should be vaccinated to ensure that a disease can not enter an epidemic phase? To answer this we reconsider the derivation of the basic reproductive number, but starting from a linearization about the point where $S = (1-V)N$ and $R = VN$, where $V \in [0,1]$ represents the initial fraction of the population immune to the disease. This results in a herd immunity threshold of

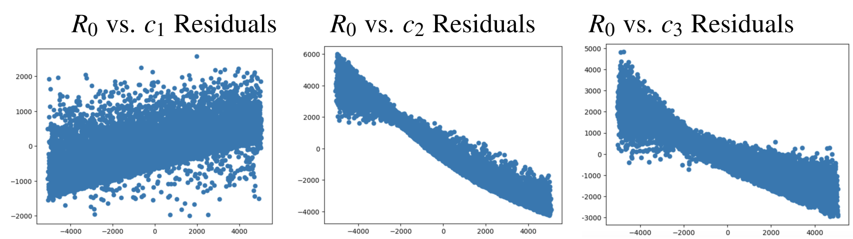
$$V = 1 - 1/R_0,$$

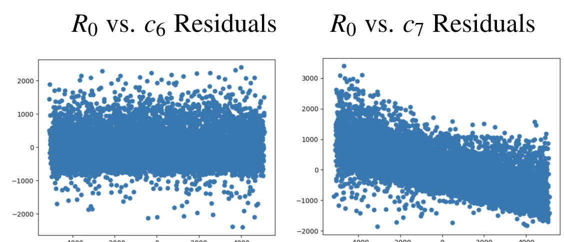
where R_0 is as computed in section 3. Using default values for parameters results in $V = 0.87$.

While it can be valuable to see the sensitivity of R_0 and i_∞ to one intervention at a time, it is perhaps more realistic to consider the combined effects of multiple interventions. We use Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficient (PRCC) analysis to gain a more global view of parameter sensitivity. We describe the LHS/PRCC approach briefly here, but a more detailed overview of can be found in [11]. Importantly, we note that we focus this analysis on the combined effect of control parameters c_1, c_2, c_3, c_6 and c_7, excluding c_4 and c_5 since their effect on R_0 and i_∞ is nonmonotonic (a necessary assumption for PRCC since it is based on linear regression).

We start with a Latin Hypercube Sampling (LHS) of 10000 values, where each parameter chosen independently and uniformly random for each parameter in the ranges described in Table 3. This sampling produces 10000 corresponding values for R_0 and i_∞. We use the analytical result for R_0 in equation (5) and use numerical solutions of system (7) to determine $i\infty$.

With this data, Partial Rank Correlation Coefficient (PRCC) values are computed to estimate the sensitivity of R_0 and i_∞ to each variable, while taking the effect of the other variables into account. Scatter plots of the subsequent residual ranking and a chart of PRCC values are shown in Figures 5, 6 and 7. In all cases, the PRCC values are statistically significant at the $p = 0.001$ level.



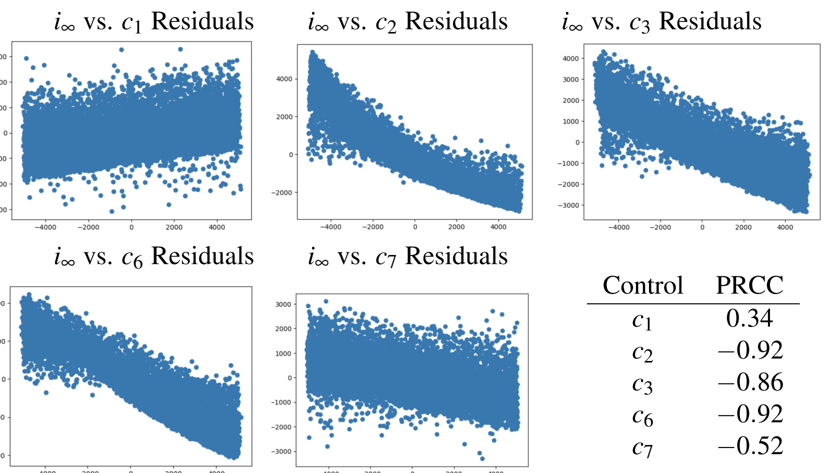


Control	PRCC
c_1	0.57
c_2	-0.98
c_3	-0.92
c_6	≈ 0
c_7	-0.69

Figure 5: LHS/PRCC results for global R_0 sensitivity.

These results show that the most effective overall interventions for reducing R_0 are c_2 (medical treatment) and c_3 (water treatment). To reduce the likelihood of an outbreak, this therefore suggests a policy that focuses resources on both reducing the time that individuals are infected as well as increasing the rate of contaminant removal. We also note that since R_0 is not dependent on c_6 the PRCC value is essentially zero and the plot of residual ranking shows no correlation.

The most effective overall interventions for reducing i_∞ are c_2 (medical treatment), c_3 (water treatment), and c_6 (vaccination). This suggests that in the endemic phase it remains important to focus on medical and water treatment, but also provides evidence for the effectiveness of implementing a vaccination routine to keep endemic infection levels as low as possible.



Control	PRCC
c_1	0.34
c_2	−0.92
c_3	−0.86
c_6	−0.92
c_7	−0.52

Figure 6: LHS/PRCC results for global i_∞ sensitivity.

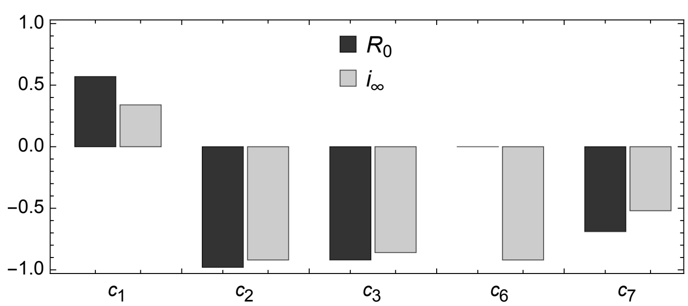


Figure 7: PRCC Values for R_0 and i_∞.

5 Discussion and Conclusion

Water-based pathogens can have the potential to spread very rapidly in a population that uses a shared water source. Some common waterborne diseases like cholera and polio are highly infectious and can spread easily throughout a population without the proper precautions. Understanding the dynamics under dual transmission pathways is an important tool in assessing the relative merits of potential intervention methods and their effects on both the epidemic and endemic phases of an outbreak.

To illustrate the effectiveness of a strategic combination of parameters, consider the scenario with $c_2 = 3$, $c_4 = 2$, and $c_5 = 0$. Here we assume a medical treatment procedure where symptomatically infected people remain infectious for approximately 3.3 days instead of 10, and a policy where people exclusively obtain drinking water from the upstream source and direct wastewater exclusively in the downstream source. This results in $R_0 = 0.76$ and $i_\infty = 0.0$, which means that under these conditions the introduction of the pathogen to a new population is unlikely to result in an epidemic and that any level of infection in the population should reduce to negligible values over time. The model here assumes a homogeneous and well-mixed population of people, which generally limits it to a relatively small scale. It is possible to generalize the approach here to include heterogeneous populations by implementing a network model. Within this framework it is also possible to further distinguish behavioral responses or interventions on individuals in subsets of populations which could lead to differences in threshold parameters or time scales in the dynamics.

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Odd Graceful Labelings of Prism Graphs

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Sage Patten graduated from Middle College at Austin Peay State University, where she contributed to this research paper during her junior and senior years of high school. Since earning her associate's degree, she has been pursuing a Bachelor of Fine Arts in Architecture with a minor in Environmental Studies at the University of Memphis, driven by a desire to blend artistic expression with mathematical principles.

Abstract

Odd graceful labelings of a graph are a variation of a graceful labeling. In each, the vertices are uniquely labeled with integers, and edges are assigned the difference between the incident vertex labels. For a graph with m edges, the goal of a graceful labeling is to have distinct edge labels 1 to m, while an odd graceful labeling has odd edge labels from 1 to $2m-1$. In this paper we construct odd graceful labelings of prism graphs, denoted $C_n \times P_2$, when n is even using the cases of $n = 6k, 6k+2$, and $6k+4$, which require similar but altered labelings.

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1 Introduction

A graceful labeling is a graph labeling that was developed over 50 years ago and which continues to be an active area of research. Graceful labelings were initially presented by Rosa [12], although he referred to them as a β-*labeling*. They were introduced to help decompose a complete graph into isomorphic subgraphs as an attempt to solve a problem known as Ringel's conjecture [11], which was made in 1963 and proven in 2021 [10]. Applications of this labeling, which was later called *graceful labeling* by Golomb [6], include other mathematical problems in coding theory and combinatorics, real-world problems involving seating arrangements, and science applications in x-ray crystallography and database management [1].

Graceful labelings are created by assigning distinct integers to vertices, with the absolute difference between these integers forming unique edge labels, while staying within a restricted set of possible labels. It is formally stated in Definition 1.1.

Definition 1. A *graceful labeling* of a graph with m edges consists of the following:

1. Vertices are assigned distinct integers between 0 and m using a function $\ell : V \to \{0, 1, \ldots, m\}$.
2. An edge xy is assigned an integer by the function $\ell : E \to \{1, 2, \ldots, m\}$ with $\ell(xy) = |\ell(x) - \ell(y)|$, resulting in a set of edge labels equal to the set $\{1, 2, \ldots, m\}$.

If a graceful labeling exists for a particular graph G, then we say G is *graceful*. Figure 1 shows that the cycle on four vertices is graceful.

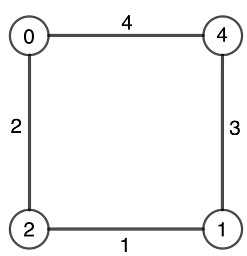

Figure 1: The cycle C_4 with a graceful labeling

Graceful labelings have been studied in numerous papers for various classes of graphs. For instance, cycles of length n, with $n \not\equiv 0$ or $3 \pmod 4$, were shown to be graceful in [12]. In addition, wheel graphs [2], pendant graphs [7], prism graphs [3], and certain sizes of generalized or stacked prism graphs (also called cylindrical grids) [8] are also graceful. Gallian's survey paper [4] provides an extensive list of other graceful graphs such as grids, helms, complete bipartite graphs, and many classes of trees. Note that it is still an open problem of whether all trees are graceful.

2 Odd Graceful Labelings

An odd graceful labeling is a variation of a graceful labeling where the first m odd integers are required as the labels of the edges. This means there must be more integer options for the vertices in order to obtain these edge labels, causing the expression $2m-1$ to replace m as the largest label on any vertex or edge. This labeling was introduced by Gnanajothi [5] and is formally stated as follows:

Definition 2. An *odd graceful labeling* of a graph with m edges consists of the following:

1. Vertices are assigned distinct integers from 0 to $2m-1$ using a function $\ell : V \to \{0,1,\ldots,2m-1\}$.
2. An edge xy is assigned an integer by the function $\ell : E \to \{1,3,\ldots,2m-1\}$ with $\ell(xy) = |\ell(x) - \ell(y)|$, resulting in a set of edge labels equal to the set $\{1,3,\ldots,2m-1\}$.

If a graph G has an odd graceful labeling, then we refer to G as being *odd graceful*. Figure 2 shows that the cycle on four vertices is odd graceful.

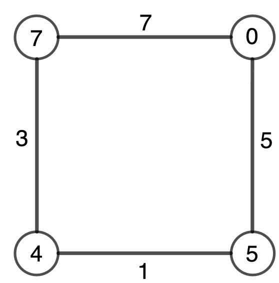

Figure 2: The cycle C_4 with an odd graceful labeling

Gnanajothi [5] proved a connection between odd graceful labelings and another graph labeling known as α-labelings, particularly that every graph with an α-labeling has an odd graceful labeling. An α-labeling [12] is a graceful labeling ℓ with an additional condition that there is an integer k such that given an edge uv, either $\ell(u) \leq k < \ell(v)$ or $\ell(v) \leq k < \ell(u)$. There are graphs, for example cycles of length $n \equiv 2 \pmod 4$, that are odd graceful, but for which an α-labeling does not exist.

Odd graceful labelings have been constructed for many classes of graphs, starting in [5] with all paths P_n, cycles C_n with even n, all complete bipartite graphs $K_{m,n}$, and any tree with up to 10 vertices. Additionally, all grid graphs were shown to have odd graceful labelings [13]. Gallian [4] provides further classes of graphs that have been shown to be odd graceful, and since graphs with α-labelings are known to also have odd graceful labelings, the known results on that type of labeling add to the list of known odd graceful graphs.

A large set of graphs are quickly eliminated from consideration for odd graceful labelings, particularly those containing an odd cycle as a subgraph. Note that this is equivalent to requiring graphs to be bipartite in order for an odd graceful labeling to possibly exist. The following was originally proven by Gnanajothi [5] and is important in limiting the cases of prism graphs that we will investigate.

Theorem 3 (Gnanajothi [5]). *A graph G is not odd graceful if it contains an odd cycle as a subgraph.*

In order to create odd edge labels, the vertex labels must alternate between even and odd integers. Following this pattern in odd cycles will inevitably result in an even difference for the final edge no matter the choice for the last vertex label.

The cycle graph, denoted C_n, has n vertices $v_1, v_2, \ldots, v_n$ as well as n edges of the form $v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $v_n v_1$. The following odd case is an immediate consequence of Theorem 3.

Corollary 4. *The cycle graph C_n is not odd graceful if n is odd.*

Gnanajothi [5] proved the following for the even case of the cycle graph. Although we do not show the full proof, we provide an odd graceful labeling for one case of an even cycle since it gives motivation for the approach we will take for prism graphs.

Theorem 5 (Gnanajothi [5])**.** *The cycle graph C_n is odd graceful if n is even.*

The following labeling function for the vertices results in an odd graceful labeling for the case of $n = 4k+2$ for some integer $k \geq 1$:

$$\ell(v_i) = \begin{cases} i-2, & i = 2,4,\ldots,4k \\ i, & i = 4k+2 \\ 8k+4-i, & i = 1,3,\ldots,2k+1 \\ 8k+2-i, & i = 2k+3, 2k+5, \ldots, 4k+1. \end{cases}$$

An example for $n = 10$, where $k = 2$, can be seen in Figure 3. Notice how the labels on the odd-indexed vertices decrease by 2 each time until $i = 2k+3 = 7$, where it drops by 4. This skipping of a vertex label is essential to make sure the last edge $v_{10}v_1$ doesn't repeat a label. To label the different cases of prism graphs in the upcoming section, we make use of similar shifts in the vertex labels to skip edge labels that are assigned at the end of the cycles.

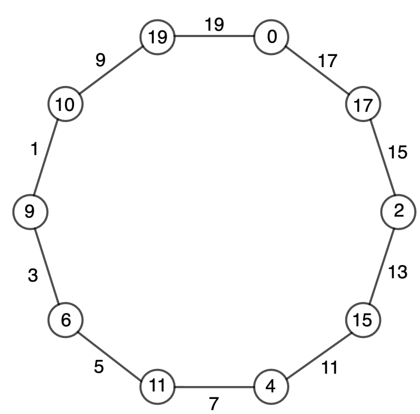

Figure 3: The cycle graph C_{10} with an odd graceful labeling

3 Prism Graphs

Prism graphs were demonstrated in [3] to have an α-labeling, and as previously discussed, all graphs with an α-labeling are odd graceful. We aim in this section to explicitly construct an odd graceful labeling for each case of prism graphs when n is even. These graphs consist of two cycles of length n with additional edges connecting each vertex on the interior cycle to a corresponding one in the exterior cycle. In the example in Figure 4, each cycle has $n = 8$ vertices.

Let $C_n \times P_2$, for integers $n \geq 3$, denote the prism graph for a cycle of length n. The product notation represents the Cartesian product of two graphs, $G \times H$. Informally, this product creates a new graph with $|V(H)|$ copies of G, with the copies of each vertex of G being connected in the manner of the edges in H. This means that for the prism graph, we get $|V(P_2)| = 2$ cycles C_n, in which we call the vertices $v_1, v_2, \ldots, v_n$ and $x_1, x_2, \ldots, x_n$. Edges are of the form $v_i v_{i+1}$ and $x_i x_{i+1}$ for integers i with $1 \leq i \leq n-1$, along with $v_1 v_n$ and $x_1 x_n$ within those two copies of the cycle, and also $v_i x_i$ with $1 \leq i \leq n$ as a path P_2 is formed between the cycles. In upcoming figures, the vertices $v_1, v_2, \ldots, v_n$ will form the interior cycle and $x_1, x_2, \ldots, x_n$ the exterior cycle.

Prism graphs have also previously been shown in [2] to have a graceful labeling for all cycle lengths n. Furthermore, using longer paths than P_2 creates the stacked prism $C_n \times P_m$ with $m \geq 2$. The stacked prism $C_n \times P_m$ was shown in [2] to be graceful for all n when m is even and some small cases of n when m is odd.

As an example, an odd graceful labeling of $C_8 \times P_2$ is shown in Figure 4. Observe that this labeling fulfills Definition 2 as the vertices are labelled distinctly with integers between 0 and 47, and the edge labels are exactly $1, 3, \ldots, 47$. Every prism graph is graceful, but we show not every prism graph is odd graceful. The next four results combine to show $C_n \times P_2$ is odd graceful if and only if n is even.

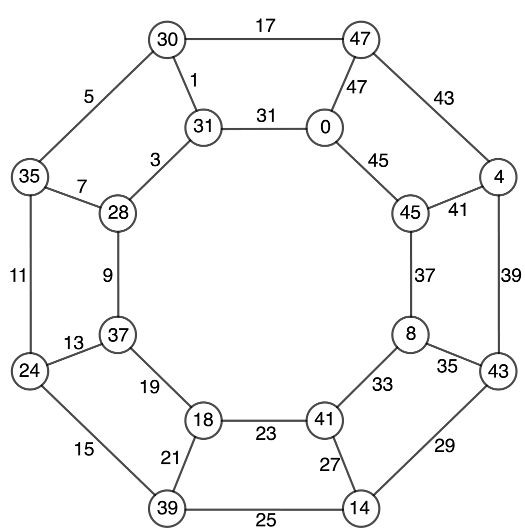

Figure 4: The prism graph $C_8 \times P_2$ with an odd graceful labeling

Corollary 6. *The prism graph $C_n \times P_2$ is not odd graceful for any odd integer $n \geq 3$.*

Proof. This is immediately true by Theorem 3 since the vertices $v_1, v_2, \ldots, v_n$ form an odd cycle. □

While the odd case of n results in the prism graph not having an odd graceful labeling, we will show how to create one for any even n. Figures 4 and 5 show odd graceful labelings of two sizes of prisms. The labeling strategy is similar initially in which the edge labels decrease by 2 as you follow v_1x_1, v_1v_2, x_1x_2, v_2x_2, x_2x_3, v_2v_3, etc. If this pattern were to continue around the prism, we would repeat edge labels when v_n and x_n connect to v_1 and x_1. We therefore have to skip two labels at specific edges to successfully construct an odd graceful labeling. The difficulty that transpires is that the edges where the skipped labels are needed occur at different locations depending on whether the even n is of the form $6k$, $6k+2$, or $6k+4$. Thus, different labeling functions are needed as we develop odd graceful labelings for each case of even-sided prism graphs. Figure 5 shows an example of the first case of an odd graceful labeling for $n = 12$, or $k = 2$.

Theorem 7. *The prism graph $C_{6k} \times P_2$ has an odd graceful labeling for all $k \geq 1$ using the following vertex labeling function:*

$$\ell(v_i) = \begin{cases} 36k+1-2i, & i = 2,4,\ldots,6k \\ 4i-4, & i = 1,3,\ldots,2k-1 \\ 4i-2, & i = 2k+1, 2k+3, \ldots, 4k-1 \\ 4i, & i = 4k+1, 4k+3, \ldots, 6k-1, \end{cases}$$

$$\ell(x_i) = \begin{cases} 36k+1-2i, & i = 1,3,\ldots,6k-1 \\ 4i-4, & i = 2,4,\ldots,2k \\ 4i-2, & i = 2k+2, 2k+4, \ldots, 4k \\ 4i, & i = 4k+2, 4k+4, \ldots, 6k. \end{cases}$$

Proof. First we observe that the number of edges in $C_{6k} \times P_2$ is $3(6k) = 18k$. Then by Definition 2, the largest edge and vertex label allowed is $2 \cdot 3n - 1 = 2(18k) - 1 = 36k - 1$. Therefore, we need to show this labeling has distinct vertex labels within the set $\{0, 1, \ldots, 36k-1\}$ with the edge labels $\ell(uv) = |\ell(u) - \ell(v)|$ being $1, 3, \ldots, 36k-1$.

Focusing first on the vertex labels, it is clear that our labels are between 0 and $36k-1$ since the largest label is $\ell(x_1) = 36k+1-2\cdot 1 = 36k-1$ and the smallest label is $\ell(v_1) = 4\cdot 1 - 4 = 0$. To show the vertex labels are distinct, we consider the sequences formed by the different lines of the two parts of our labeling function. The first line of $\ell(x_i)$ is the sequence of odd integers

$$36k-1, 36k-5, \ldots, 24k+3$$

since $36k+1-2(6k-1) = 24k+3$. Then the first line of $\ell(v_i)$ is the sequence

$$36k-3, 36k-7, \ldots, 24k+1.$$

We see that these two sequences consist of distinct odd integers.

The remaining labels will all be even integers, starting with the second line of $\ell(v_i)$:

$$0, 8, \ldots, 8k-8$$

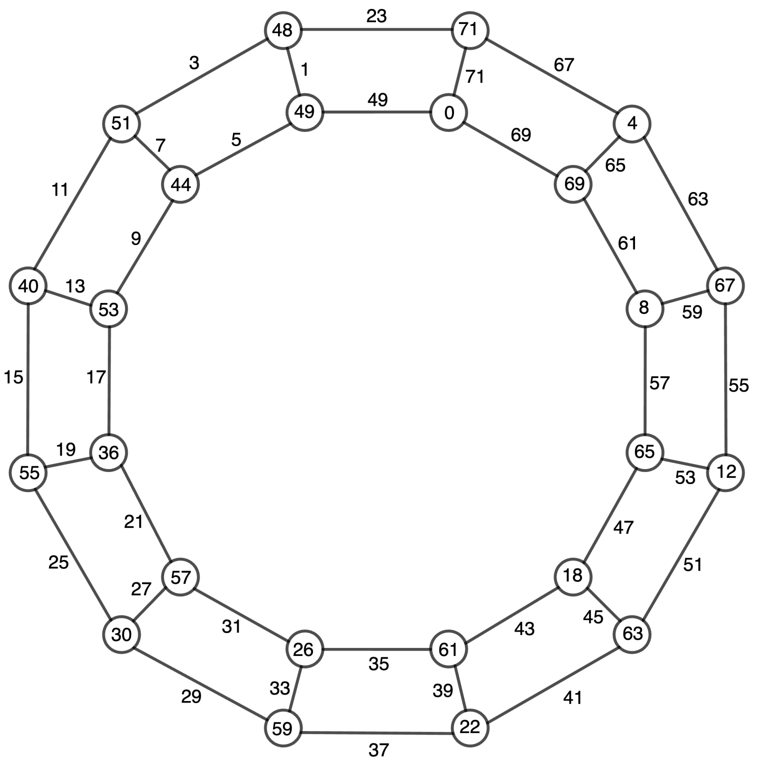

Figure 5: The prism graph $C_{12} \times P_2$ with an odd graceful labeling

since $4(2k-1)-4 = 8k-8$, while the second line of $\ell(x_i)$ is the following:

$$4, 12, \ldots, 8k-4.$$

Note that these two sequences have k entries, so for the example in Figure 5 where $k = 2$, the sequences are merely $0, 8$ and $4, 12$. We list three terms of each labeling sequence to demonstrate the difference between terms, but keep in mind some condense down to only 1 or 2 terms when k is small.

We continue by observing that the third line of $\ell(v_i)$ is the sequence

$$8k+2, 8k+10, \ldots, 16k-6$$

by simplifying $4(2k+1)-2, 4(2k+3)-2, \ldots, 4(4k-1)-2$. The third line of $\ell(x_i)$ is the sequence

$$8k+6, 8k+14, \ldots, 16k-2.$$

The fourth line of $\ell(v_i)$ is the sequence

$$16k+4, 16k+12, \ldots, 24k-4.$$

Finally, the fourth line of $\ell(x_i)$ is the sequence

$$16k+8, 16k+16, \ldots, 24k.$$

From those six lines of our function, one can see that they are nonoverlapping sequences of even labels. Therefore, all of our vertex labels are distinct integers in $\{0,1,\ldots,36k-1\}$, which satisfies the first part of the definition of an odd graceful labeling.

We now consider the set of labels on the edges, where we need to show this set is $\{1,3,5,\ldots,36k-1\}$. We will examine sequences of labels for specific subsets of edges whose incident vertex pairs use the same lines of our vertex labeling function. First consider a set of edges connecting vertices between the two cycles: $v_1x_1, v_2x_2, \ldots, v_{2k}x_{2k}$, stopping just before the first shift in vertex labels between the second and third lines of the labeling function. For odd i, combining line 2 of $\ell(v_i)$ and line 1 of $\ell(x_i)$,

$$\begin{aligned}\ell(v_ix_i) &= |(4i-4)-(36k+1-2i)| \\ &= |6i-5-36k| \\ &= 36k-6i+5.\end{aligned}$$

For even i, notice that $\ell(v_i)=\ell(x_{i-1})$ and $\ell(x_i)=\ell(v_{i-1})$. Therefore, we get the same result for the edge label after applying the absolute value to the difference. In this particular case, $\ell(v_ix_i)=|(36k+1-2i)-(4i-4)|=36k-6i+5$. The edge labels $\ell(v_ix_i)$ for $1\le i\le 2k$ form the sequence

$$36k-1, 36k-7, \ldots, 24k+5. \tag{1}$$

Next, we consider the edges, $v_1v_2, x_2x_3, v_3v_4, \ldots, v_{2k-1}v_{2k}, x_{2k}x_{2k+1}$ that alternate between the two cycles. For odd i, we obtain the following from line 2 of $\ell(v_i)$ and line 1 of $\ell(v_i)$ with $i+1$ substituted for i:

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(4i-4)-[36k+1-2(i+1)]| \\ &= |6i-36k-3| \\ &= 36k-6i+3.\end{aligned}$$

Observe that for even i, $\ell(x_ix_{i+1})$ results in the same expression as above. For integers i with $1\le i\le 2k$, this develops the sequence of labels

$$36k-3, 36k-9, \ldots, 24k+3. \tag{2}$$

Then we consider the edges $x_1x_2, v_2v_3, x_3x_4, \ldots, v_{2k-2}v_{2k-1}, x_{2k-1}x_{2k}$ with the opposite alternating pattern. Using lines 1 and 2 from $\ell(x_i)$, we have for odd i

$$\begin{aligned}\ell(x_ix_{i+1}) &= |(36k+1-2i)-[4(i+1)-4]| \\ &= 36k-6i+1,\end{aligned}$$

which is identical for $\ell(v_iv_{i+1})$ when i is even. This results in the following sequence as $i=1$ to $2k-1$:

$$36k-5, 36k-11, \ldots, 24k+7. \tag{3}$$

Observe that the sequences in (1), (2), and (3) do not overlap and include every odd integer $24k+3$ to $36k-1$.

Now we consider three similar sets of edges before the last shift in vertex labels, starting with $v_{2k+1}x_{2k+1}, v_{2k+2}x_{2k+2}, \ldots, v_{4k}x_{4k}$. For odd i,

$$\begin{aligned}\ell(v_ix_i) &= |(36k+1-2i)-(4i-2)| \\ &= 36k-6i+3.\end{aligned}$$

The same expression is obtained in the even i case because of the absolute value. This results, as $i = 2k+1$ to $4k$, in the sequence of labels

$$24k-3, 24k-9, \dots, 12k+3. \tag{4}$$

Next, we consider the edges $v_{2k+1}v_{2k+2}, x_{2k+2}x_{2k+3}, v_{2k+3}v_{2k+4}, \dots, v_{4k-1}v_{4k}, x_{4k}x_{4k+1}$, which have the labels

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(4i-2) - [36k+1-2(i+1)]|\\ &= 36k-6i+1,\end{aligned}$$

and likewise for $\ell(x_ix_{i+1})$. This develops the following sequence of labels as $i = 2k+1$ to $4k$:

$$24k-5, 24k-11, \dots, 12k+1. \tag{5}$$

Next, the labels on edges $v_{2k}v_{2k+1}, x_{2k+1}x_{2k+2}, v_{2k+2}v_{2k+3}, \dots, v_{4k-2}v_{4k-1}, x_{4k-1}x_{4k}$ have the formula

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(36k+1-2i) - [4(i+1)-2]|\\ &= 36k-6i-1,\end{aligned}$$

which matches each $\ell(x_ix_{i+1})$. As we substitute $i = 2k$ to $4k-1$, the sequence that develops is

$$24k-1, 24k-7, \dots, 12k+5. \tag{6}$$

One can see that the sequences in (4), (5), and (6) include the distinct odd integers from $12k+1$ to $24k-1$. However, note the label $24k+1$ has been skipped thus far.

Now we consider the edges $v_{4k+1}x_{4k+1}, v_{4k+2}x_{4k+2}, \dots, v_{6k}x_{6k}$. For odd i,

$$\begin{aligned}\ell(v_ix_i) &= |(36k+1-2i) - 4i|\\ &= 36k-6i+1,\end{aligned}$$

and we get the same formula for even i. This results in the following sequence of labels as $i = 4k+1$ to $6k$:

$$12k-5, 12k-11, \dots, 1. \tag{7}$$

Next, we consider the edges $v_{4k}v_{4k+1}, x_{4k+1}x_{4k+2}, v_{4k+2}v_{4k+3}, \dots, v_{6k-2}v_{6k-1}, x_{6k-1}x_{6k}$, which have the labels

$$\begin{aligned}\ell(v_iv_{i+1}) &= |(36k+1-2i) - [4(i+1)]|\\ &= 36k-6i-3.\end{aligned}$$

The same expression results from calculating $\ell(x_ix_{i+1})$, which as $i = 4k$ to $6k-1$ develops the sequence of labels

$$12k-3, 12k-9, \dots, 3. \tag{8}$$

Finally, the edges $v_{4k+1}v_{4k+2}, x_{4k+2}x_{4k+3}, v_{4k+3}v_{4k+4}, \dots, x_{6k-2}x_{6k-1}, v_{6k-1}v_{6k}$ are assigned

$$\begin{aligned}\ell(v_iv_{i+1}) &= |4i - [36k+1-2(i+1)]|\\ &= 36k-6i-1.\end{aligned}$$

For $4k+1 \le i \le 6k-1$, the labeling sequence is

$$12k-7, 12k-13, \dots, 5. \tag{9}$$

The sequences in (7), (8), and (9) include every odd integer 1 to $12k-3$. Note that a second label was skipped, particularly $12k-1$.

The final edge labels are

$$\ell(v_{6k}v_1) = |[36k+1-2(6k)] - (4\cdot 1-4)| = 24k+1$$

and

$$\ell(x_{6k}x_1) = |4(6k) - (36k + 1 - 2 \cdot 1)| = |24k - 36k + 1| = 12k - 1.$$

Since these are exactly the two labels that were skipped between the previous sequences, we have shown that the set of edge labels includes the entire sequence of odd integers from 1 to $36k - 1$. Thus, we have proven that this is an odd graceful labeling when the length of the cycle is $6k$. □

We now proceed to the case of the cycle length n being of the form $6k+2$, an example of which was in Figure 4 when $n = 8$.

Theorem 8. *The prism graph $C_{6k+2} \times P_2$ has an odd graceful labeling for all $k \geq 1$ using the following vertex labeling function:*

$$\ell(v_i) = \begin{cases} 36k + 13 - 2i, & i = 2, 4, \ldots, 6k \\ 4i - 1, & i = 6k + 2 \\ 4i - 4, & i = 1, 3, \ldots, 2k + 1 \\ 4i - 2, & i = 2k + 3, 2k + 5, \ldots, 4k + 1 \\ 4i, & i = 4k + 3, 4k + 5, \ldots, 6k + 1, \end{cases}$$

$$\ell(x_i) = \begin{cases} 36k + 13 - 2i, & i = 1, 3, \ldots, 6k + 1 \\ 4i - 4, & i = 2, 4, \ldots, 2k \\ 4i - 2, & i = 2k + 2, 2k + 4, \ldots, 4k \\ 4i, & i = 4k + 2, 4k + 4, \ldots, 6k \\ 4i - 2, & i = 6k + 2. \end{cases}$$

Proof. First note that with $3n = 3(6k+2) = 18k+6$ edges, our largest edge and vertex label is $2(18k+6) - 1 = 36k+11$. Therefore, we first show that our vertex labels are distinct integers in the set $\{0, 1, \ldots, 36k+11\}$. It is clear that our labels are within this set since the largest label is $\ell(x_1) = 36k + 13 - 2 \cdot 1 = 36k + 11$ and the smallest label is $\ell(v_1) = 4 \cdot 1 - 4 = 0$.

As in the $n = 6k$ result, to show the vertex labels are distinct, we consider the sequences formed by the different lines of the two parts of our labeling function. Table 1 shows the sequences of vertex labels, with the top rows containing the odd labels and the bottom portion displaying the even labels. One can observe that all vertices are included and the labeling sequences are nonoverlapping. Therefore, all of our vertex labels are distinct integers in $\{0, 1, \ldots, 36k+11\}$, which satisfies the first part of the definition of an odd graceful labeling.

We now consider the labels on the edges, where we need to show this set is exactly the distinct odd integers $1, 3, 5, \ldots, 36k+11$. For brevity, we will omit the details of obtaining the labeling sequences, as the differences in vertex labels are calculated similarly to those in the $n = 6k$ result. Table 2 shows the edge and labeling sequences.

Observe that all odd integers between 7 and $36k+11$ are included within the edge labels in Table 2 with the exception of $12k+5$ and $24k+7$. The remaining edge labels involve at least one of the separately labeled vertices v_{6k+2} or x_{6k+2}: $\ell(x_{6k+1}x_{6k+2}) = 5, \ell(v_{6k+1}v_{6k+2}) = 3$, $\ell(v_{6k+2}x_{6k+2}) = 1, \ell(v_{6k+2}v_1) = 24k+7$, and $\ell(x_{6k+2}x_1) = 12k+5$.

Vertex sequence	Labeling sequence
$x_1, x_3, \ldots, x_{6k+1}$	$36k+11, 36k+7, \ldots, 24k+11$
$v_2, v_4, \ldots, v_{6k}$	$36k+9, 36k+5, \ldots, 24k+13$
v_{6k+2}	$24k+7$
$v_1, v_3, \ldots, v_{2k+1}$	$0, 8, \ldots, 8k$
$x_2, x_4, \ldots, x_{2k}$	$4, 12, \ldots, 8k-4$
$x_{2k+2}, x_{2k+4}, \ldots, x_{4k}$	$8k+6, 8k+14, \ldots, 16k-2$
$v_{2k+3}, v_{2k+5}, \ldots, v_{4k+1}$	$8k+10, 8k+18, \ldots, 16k+2$
$x_{4k+2}, x_{4k+4}, \ldots, x_{6k}$	$16k+8, 16k+16, \ldots, 24k$
$v_{4k+3}, v_{4k+5}, \ldots, v_{6k+1}$	$16k+12, 16k+20, \ldots, 24k+4$
x_{6k+2}	$24k+6$

Table 1: The vertex labels for the prism graph $C_{6k+2} \times P_2$

These final five edge labels complete the entire sequence of odd integers from 1 to $36k+11$, include filling in the two skipped labels of $12k+5$ and $24k+7$. Thus, we have shown this is odd graceful labeling for when the cycle has length $6k+2$.

□

We conclude our examination of prism graphs with cycle lengths in which $n = 6k+4$. Figure 6 shows an example from this case of an odd graceful labeling of $C_{16} \times P_2$.

Theorem 9. *The prism graph $C_{6k+4} \times P_2$ has an odd graceful labeling for all $k \geq 0$ using the following vertex labeling function:*

$$\ell(v_i) = \begin{cases} 36k+25-2i, & i = 2, 4, \ldots, 2k \\ 36k+23-2i, & i = 2k+2, 2k+4, \ldots, 6k+2 \\ 4i-3, & i = 6k+4 \\ 4i-4, & i = 1, 3, \ldots, 4k+1 \\ 4i-2, & i = 4k+3, 4k+5, \ldots, 6k+3, \end{cases}$$

$$\ell(x_i) = \begin{cases} 36k+25-2i, & i = 1, 3, \ldots, 2k+1 \\ 36k+23-2i, & i = 2k+3, 2k+5, \ldots, 6k+3 \\ 4i-4, & i = 2, 4, \ldots, 2k \\ 4i-6, & i = 2k+2 \\ 4i-4, & i = 2k+4, 2k+6, \ldots, 4k+2 \\ 4i-2, & i = 4k+4, 4k+6, \ldots, 6k+2 \\ 4i-4, & i = 6k+4. \end{cases}$$

Proof. First, note that we have the situation of $k = 0$ included whereas previous results began with $k = 1$. This will result in some rows of the upcoming table being empty sequences; however, the labeling will be odd graceful in that single case as well.

Edge sequence	Labeling sequence
$v_1x_1, v_2x_2, \ldots, v_{2k+1}x_{2k+1}$	$36k+11, 36k+5, \ldots, 24k+11$
$v_1v_2, x_2x_3, v_3v_4, \ldots, x_{2k}x_{2k+1}, v_{2k+1}v_{2k+2}$	$36k+9, 36k+3, \ldots, 24k+9$
$x_1x_2, v_2v_3, x_3x_4, \ldots, x_{2k-1}x_{2k}, v_{2k}v_{2k+1}$	$36k+7, 36k+1, \ldots, 24k+13$
$v_{2k+2}x_{2k+2}, v_{2k+3}x_{2k+3}, \ldots, v_{4k+1}x_{4k+1}$	$24k+3, 24k-3, \ldots, 12k+9$
$x_{2k+1}x_{2k+2}, v_{2k+2}v_{2k+3}, x_{2k+3}x_{2k+4}, \ldots, x_{4k-1}x_{4k}, v_{4k}v_{4k+1}$	$24k+5, 24k-1, \ldots, 12k+11$
$x_{2k+2}x_{2k+3}, v_{2k+3}v_{2k+4}, x_{2k+4}x_{2k+5}, \ldots, x_{4k}x_{4k+1}, v_{4k+1}v_{4k+2}$	$24k+1, 24k-5, \ldots, 12k+7$
$v_{4k+2}x_{4k+2}, v_{4k+3}x_{4k+3}, \ldots, v_{6k+1}x_{6k+1}$	$12k+1, 12k-5, \ldots, 7$
$x_{4k+1}x_{4k+2}, v_{4k+2}v_{4k+3}, \ldots, v_{6k}v_{6k+1}$	$12k+3, 12k-3, \ldots, 9$
$x_{4k+2}x_{4k+3}, v_{4k+3}v_{4k+4}, x_{4k+3}x_{4k+4}, \ldots, v_{6k-1}v_{6k}, x_{6k}x_{6k+1}$	$12k-1, 12k-7, \ldots, 11$

Table 2: The edge labels for the prism graph $C_{6k+2} \times P_2$

With $3n = 3(6k+4) = 18k+12$ edges, the largest edge and vertex label would be $2(18k+12)-1 = 36k+23$. Then we first show that our vertex labels are distinct integers in $\{0,1,\ldots,36k+23\}$. The largest label is $\ell(x_1) = 36k+25-2\cdot 1 = 36k+23$ and the smallest label is $\ell(v_1) = 4\cdot 1 - 4 = 0$, so the vertex labels are all within the set.

Following the a similar process as the previous two cases, Table 3 shows the sequences of odd and even vertex labels. One can observe that all of our vertex labels are distinct integers in $\{0,1,\ldots,36k+23\}$, satisfying the first part of the definition of an odd graceful labeling.

Vertex Sequence	Labeling Sequence
$x_1, x_3, \ldots, x_{2k+1}$	$36k+23, 36k+19, \ldots, 32k+23$
$x_{2k+3}, x_{2k+5}, \ldots, x_{6k+3}$	$32k+17, 32k+13, \ldots, 24k+17$
$v_2, v_4, \ldots, v_{2k}$	$36k+21, 36k+17, \ldots, 32k+25$
$v_{2k+2}, v_{2k+4}, \ldots, v_{6k+2}$	$32k+19, 32k+15, \ldots, 24k+19$
v_{6k+4}	$24k+13$
$v_1, v_3, \ldots, v_{4k+1}$	$0, 8, \ldots, 16k$
$x_2, x_4, \ldots, x_{2k}$	$4, 12, \ldots, 8k-4$
$x_{2k+4}, x_{2k+6}, \ldots, x_{4k+2}$	$8k+12, 8k+20, \ldots, 16k+4$
$v_{4k+3}, v_{4k+5}, \ldots, v_{6k+3}$	$16k+10, 16k+18, \ldots, 24k+10$
$x_{4k+4}, x_{4k+6}, \ldots, x_{6k+2}$	$16k+14, 16k+22, \ldots, 24k+6$
x_{2k+2}	$8k+2$
x_{6k+4}	$24k+12$

Table 3: The vertex labels for the prism graph $C_{6k+4} \times P_2$

We now consider the labels on the edges. We need to show the set of edge labels is $\{1,3,5,\ldots,36k+23\}$. Table 4 shows the edge and labeling sequences, which are

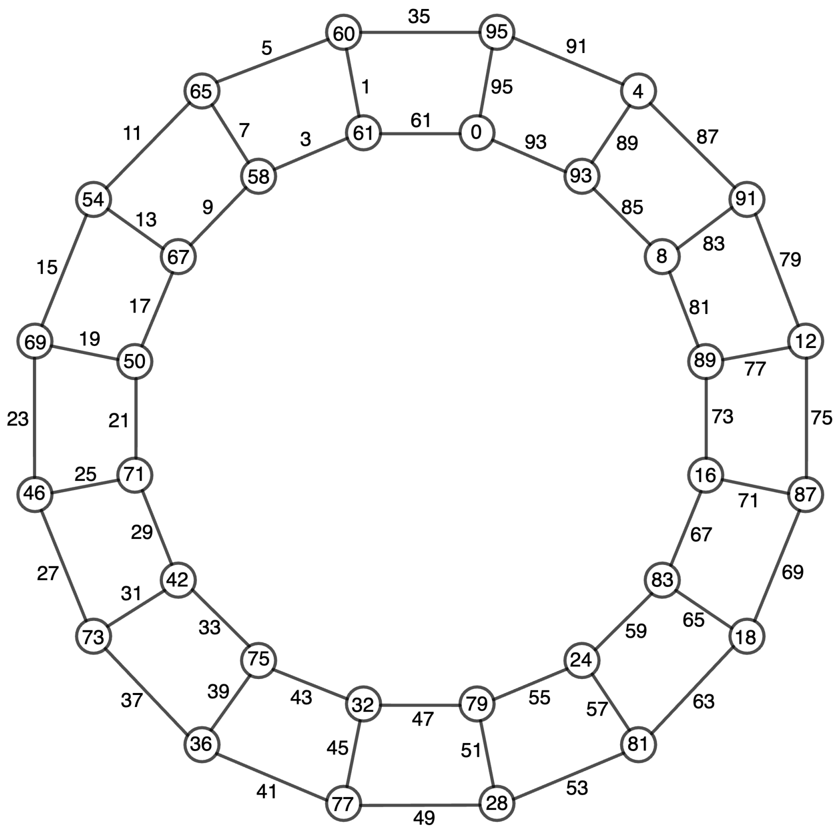

Figure 6: The prism graph $C_{16} \times P_2$ with an odd graceful labeling

calculated similarly to cases $n = 6k$ and $n = 6k+2$. This case does differ, however, in that we next have to consider some individual edges as shifts occur within our labeling function:

$$\ell(x_{2k+1}x_{2k+2}) = |36k+25-2(2k+1)-[4(2k+2)-6]| = 24k+21,$$
$$\ell(v_{2k+1}v_{2k+2}) = |[4(2k+1)-4)]-[36k+23-2(2k+2)]| = 24k+19,$$
$$\ell(v_{2k+2}x_{2k+2}) = |[36k+23-2(2k+2)]-[4(2k+2)-6]| = 24k+17,$$
$$\ell(x_{2k+2}x_{2k+3}) = |[4(2k+2)-6)]-[36k+23-2(2k+3)]| = 24k+15.$$

At this point, observe that Table 4 and the previous four labels include every odd integer 7 to $36k+23$, with the exception of $24k+13$ and $12k+11$ being skipped. There are five edges remaining that are incident on at least one of v_{6k+4} or x_{6k+4}, which are labeled by $24k+13$ and $24k+12$, respectively. They are the following: $\ell(x_{6k+3}x_{6k+4}) = 5$, $\ell(v_{6k+3}v_{6k+4}) = 3$, $\ell(v_{6k+4}x_{6k+4}) = 1$, $\ell(v_{6k+4}v_1) = 24k+13$, and $\ell(x_{6k+4}x_1) = 12k+11$.

We have shown that the set of edge labels includes the entire sequence of odd integers from 1 to $36k+23$. Thus, we have shown that this is an odd graceful labeling when the cycle length is of the form $6k+4$. □

Edge sequence	Labeling sequence
$v_1x_1, v_2x_2, \ldots, v_{2k+1}x_{2k+1}$	$36k+23, 36k+17, \ldots, 24k+23$
$x_1x_2, v_2v_3, x_3x_4, \ldots, x_{2k-1}x_{2k}, v_{2k}v_{2k+1}$	$36k+19, 36k+13, \ldots, 24k+25$
$v_1v_2, x_2x_3, v_3v_4, \ldots, v_{2k-1}v_{2k}, x_{2k}x_{2k+1}$	$36k+21, 36k+15, \ldots, 24k+27$
$v_{2k+3}x_{2k+3}, v_{2k+4}x_{2k+4}, \ldots, v_{4k+2}x_{4k+2}$	$24k+9, 24k+3, \ldots, 12k+15$
$v_{2k+2}v_{2k+3}, x_{2k+3}x_{2k+4}, v_{2k+4}v_{2k+5}, \ldots, v_{4k}v_{4k+1}, x_{4k+1}x_{4k+2}$	$24k+11, 24k+5, \ldots, 12k+17$
$v_{2k+3}v_{2k+4}, x_{2k+4}x_{2k+5}, v_{2k+5}v_{2k+6}, \ldots, v_{4k+1}v_{4k+2}, x_{4k+2}x_{4k+3}$	$24k+7, 24k+1, \ldots, 12k+13$
$v_{4k+3}x_{4k+3}, v_{4k+4}x_{4k+4}, \ldots, v_{6k+3}x_{6k+3}$	$12k+7, 12k+1, \ldots, 7$
$v_{4k+3}v_{4k+4}, x_{4k+4}x_{4k+5}, v_{4k+5}v_{4k+6}, \ldots, v_{6k+1}v_{6k+2}, x_{6k+2}x_{6k+3}$	$12k+5, 12k-1, \ldots, 11$
$v_{4k+2}v_{4k+3}, x_{4k+3}x_{4k+4}, v_{4k+4}v_{4k+5}, \ldots, x_{6k+1}x_{6k+2}, v_{6k+2}v_{6k+3}$	$12k+9, 12k+3, \ldots, 9$

Table 4: The edge labels for the prism graph $C_{6k+4} \times P_2$

4 Open Problems

A natural extension of cycle graphs (which could be viewed as $C_n \times P_1$) and prism graphs ($C_n \times P_2$) would be to increase the value m on $C_n \times P_m$, referred to as the stacked prism or cylindrical grid graph. Observe that as long as n is even, all cycles within the graph would be of even length. Note that although various cases of $C_n \times P_m$ have been shown to be graceful (see Table 1 in [4]), it appears to remain an open problem for whether that is true for all m and n. From an α-labeling perspective, labelings were developed in [9] for all cases of even n except for $C_{4k+2} \times P_{2\ell+1}$, which remains open.

Another way to view prism graphs is as the most basic version of a generalized Petersen graph, $GP(n,k)$ with $k < n/2$. Like prism graphs, $GP(n,k)$ has vertices $v_1, v_2, \ldots, v_n$ and $x_1, x_2, \ldots, x_n$. It also has edges of the form v_ix_i from 1 to n and x_ix_{i+1} for $1 \le i \le n-1$ along with x_nx_1. Where it differs are the edges between the interior v_i vertices. The prism graph, which is the case $GP(n,1)$, has edges v_iv_{i+1}, but the more general $GP(n,k)$ has edges v_iv_{i+k}. Similar to stacked prisms, progress has been made demonstrating $GP(n,k)$ is graceful for small n and any $k < n/2$, but the question for larger n and k remains unresolved for both graceful and odd graceful labelings.

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Generalized Bondage Number: The k-synchronous bondage number of a graph

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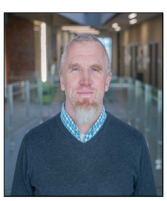

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Abstract

We investigate a generalization of the bondage number of a graph called the k-*synchronous bondage number*. The k-synchronous bondage number of a graph is the smallest number of edges that, when removed, increases the domination number by k. In this paper, we discuss the 2-synchronous bondage number and then generalize to the k-synchronous bondage number. We present k-synchronous bondage numbers for several graph classes and give bounds for general graphs. We propose this characteristic as a metric of the connectivity of a simple graph with possible uses in the field of network design and optimization.

1 Introduction

Graphs serve as a mathematical tool for analyzing networks where the vertices of graphs can represent stations, transmitters, people, computers, cell phones, or cities while the edges demonstrate the connections between these objects, such as railroads, power lines, friendships, computer connections, signals, or roads. Representing such networks as graphs allows us to apply the tools and properties of graph theory to real world problems thereby providing rigorously proven solutions.

An unreliable network is one that can be easily disrupted either maliciously or accidentally. To ensure the reliability of a network, an understanding of its purpose and sensitivities must be taken into account throughout its construction and preservation. Thus, we must determine the minimum requirements in order for a network to remain operational as well as which parts of the network can break down and thus cause a failure state.

When modeling networks with graphs, we can analyze different failure states and, therefore, quantify network reliability. For instance, if the network is operational as long as the graph is connected, then the graph is in a failure state once there are two or more components. Then we can evaluate the network's strength in comparison to other potential network designs by counting the minimum number of edges which must be removed in order to render the network inoperable. This measure of connectivity, known as the *edge connectivity* γ was first introduced by Beineke and Harary [3] as the minimum number of edges whose removal results in a disconnected graph. Sampathkum [10] extended this idea of connectivity to *g-component edge connectivity* which is the minimum number of edges that must be removed in order to create at least g components.

Connectivity and number of components are not the only properties that may be considered. As an example, the *component order edge connectivity* also extends

Beineke and Harary's idea by considering the removal of edges until every component of the graph has order less than a given positive integer k (see [4], [6], and [7]).

In this paper we will focus on edge removal and the impact on domination number of a graph. The *domination number* of a graph is the cardinality of the smallest set of vertices, V', so that every vertex not in V' is adjacent to at least one vertex in V'. The minimum number of edge removals that increases the domination number by one was first considered in [2] and called *dominating line-stability*. Later, in [5], the authors define this same concept to be the *bondage number* of a graph. Here we expand upon the concept of bondage number by investigating edge removals that increase the domination number by a specified number, k. We consider the graph, and hence the network it represents, to be in a failure state when the domination number increases by k. In order to achieve a failure state we are looking for an existential quantity rather than a universal one. That is to say, we are seeking the smallest positive integer m such that there exists an edge set of cardinality m which, when removed, creates a failure state-as opposed to finding the smallest positive integer M such that all edge sets of cardinality M, when removed, will create a failure state. Hence, for our purposes, if we remove any edge set with fewer than m edges, the network will be operational.

2 Background and Definitions

For our purposes, all graphs are assumed to be undirected simple graphs; that is, graphs with no loops and in which no two vertices share more than a single edge between them. If $G=(V,E)$ and $S \subseteq V$, then the *induced subgraph on S*, denoted $G[S]$, is the subgraph of G with vertex set S which includes all the edges in G whose vertices are in S. If $H=(V',E')$ we will denote the *disjoint union* of G and H as $G \oplus H = (V \cup V', E \cup E')$. For other graph theory notation and terminology we will use [12].

Given a graph $G=(V,E)$, a set of vertices $D \subseteq V$ is a *dominating set* if all vertices in V are either in D or adjacent to a vertex in D. The *domination number* of the graph, $\gamma(G)$, is the minimum size of a dominating set in G. The study of this characteristic is summarized efficiently in [8].

In 1990, Fink et al.[5] introduced, in its modern form, the idea of *bondage number*. The bondage number of a graph, denoted $b(G)$, is the size of the smallest subset of edges of G, which, when removed, increases the domination number. In 2013, Xu [13] defines a *bondage set* as any $\mathscr{E} \subseteq E$ such that $\gamma(G-\mathscr{E}) > \gamma(G)$. Furthermore a *minimum bondage set* is a bondage set with the smallest possible cardinality. We also know if $\mathscr{E}$ is a minimum bondage set, then $\gamma(G-\mathscr{E}) = \gamma(G)+1$ because the removal of any single edge can increase the domination number by at most one, and therefore, we can remove single edges until γ increases by exactly 1.

Given a set of graphs $\mathscr{G}$, we define the *minimum bondage number of* $\mathscr{G}$ as $b(\mathscr{G}) = \min\{b(G) : G \in \mathscr{G}\}$.

Given a graph $G=(V,E)$, we define the *bondage graphs of* G, denoted $\mathrm{BG}(G)$, to be the set of all graphs $G' = G-\mathscr{E}$ for some bondage set $\mathscr{E} \subseteq E$.

Similarly, given a graph $G=(V,E)$, we define the *minimum bondage graphs of* G, denoted $\mathrm{MBG}(G)$, to be the set of all graphs $G' = G-\mathscr{E}$ for any minimum bondage set $\mathscr{E} \subseteq E$.

We will say that G' is the result of a *bondage move* if $G' \in \mathrm{BG}(G)$. So, a bondage move is the removal of a bondage set from G and we will say the *size of the bondage move* is the size of the bondage set. Similarly, we will say that G' is the result of a *minimum bondage move* if $G' \in \mathrm{MBG}(G)$ and we will say the *size of a minimum bondage move* is the size of a minimum bondage set. A minimum bondage move is, therefore, the removal of a minimum bondage set from G.

So, if we are completing iterative bondage moves on some graph G, then $b(\mathrm{MBG}(G))$ is the minimum bondage number over the set of graphs resulting from all possible first bondage moves on G.

In this paper, we define *k-synchronous bondage set*, *minimum k-synchronous bondage set*, and *k-synchronous bondage number* which generalize a bondage set, minimum bondage set, and bondage number, respectively, by increasing the domination number by k. For k-synchronous bondage, we are removing a subset of edges simultaneously. Note that we can increase the domination number by making k minimum bondage moves which will result in a domination number of $\gamma(G)+k$, however, this will not always correspond to a minimum k-synchronous bondage set. Throughout this paper, we will assume that $|V| \geq k+\gamma(G)$ since the domination number cannot be larger than the order of the graph.

Definition 1 (k-synchronous bondage set)**.** Given a graph $G=(V,E)$ and a positive integer k, a set $\mathscr{E} \subseteq E$ is a k-synchronous bondage set if $\gamma(G-\mathscr{E}) = \gamma(G)+k$.

So a set of edges is a k-synchronous bondage set if the removal of the edges increases the domination number by k.

Definition 2 (minimum k-synchronous bondage set)**.** Given a graph $G=(V,E)$, a set $\mathscr{E} \subseteq E$ is a minimum k-synchronous bondage set if $\gamma(G-\mathscr{E}) = \gamma(G)+k$ and for all $E' \subseteq E$ with $|E'| < |\mathscr{E}|$, $\gamma(G-E') < \gamma(G)+k$.

Therefore, a minimum k-synchronous bondage set is a subset of the edge set whose removal increases the domination number by k and there does not exist a smaller subset that results in the domination number increasing by k.

Definition 3 (k-synchronous bondage number)**.** Given a graph $G=(V,E)$ and a positive integer k, the k-synchronous bondage number of G, denoted $Sb_k(G)$, is the size of a minimum k-synchronous bondage set.

Thus, the k-synchronous bondage number of G is the minimum number of edges that can be removed so that the domination number increases by k.

Connecting the previous definitions with bondage number, we see that the study of Sb_1 would be the same as the study of the bondage number, as indicated in the following proposition.

Proposition 4. *For any graph G, $b(G) = Sb_1(G)$.*

However, the k-synchronous bondage number differs from previous studies on $b(G)$ whenever $k \geq 1$. Our approach is novel in that we can specify any desired increase by setting the value of k. The combined size of two successive minimum bondage moves serves only as an upper bound for Sb_2, so it is essential to discuss several concepts regarding the bondage number for $k \geq 2$.

Note that for any graph G and edge e, $\gamma(G) \leq \gamma(G-e) \leq \gamma(G)+1$. This immediately implies the following proposition.

Proposition 5. *For any graph G and any positive integer k,*

$$Sb_k(G) \geq Sb_{k-1}(G) + 1 \geq k.$$

In this paper, we consider 2-synchronous bondage in section 3 where we analyze the relationship between 2-synchronous bondage and consecutive bondage moves. We also provide some bounds specific to 2-synchronous bondage for general graphs. In section 4, we proceed to demonstrate several properties of k-synchronous bondage numbers and provide proofs for the k-synchronous bondage of paths, cycles, trees and complete graphs. We conclude by presenting additional areas of potential research into the value of k-synchronous bondage numbers.

3 Properties of Sb_2

The value of studying Sb_k is two-fold. First, suppose we obtain $Sb_k(G)$ for some graph $G = (V, E)$ by iterative minimum bondage moves. This method will require us to analyze graphs outside of the families typically considered when studying $b(G)$. Beyond this, however, we find that for some graphs, G, $Sb_k(G)$ is less than the summation of the size of k iterative minimum bondage moves. For example, consider graph H in Figure 1 which consists of a complete graph on a set $A = \{a, b, c, d\}$, a complete graph on a set $B = \{e, f, g, h\}$ with the edge eh removed, a complete graph on a set $C = \{i, j, k, l\}$ with the edges ik and jl removed, along with the edges $E' = \{de, hi\}$, and f is a vertex of a complete graph on the set $F = \{f, f_2, f_3, \ldots, f_n\}$ with $n \geq 5$, and g is a vertex of a complete graph on $G = \{g, g_2, g_3, \ldots, g_m\}$ with $m \geq 5$. The induced subgraphs on F and G are complete subgraphs denoted by the dashed circles in Figure 1.

Notice $\gamma(H) = 5$ and a minimum dominating set is $D = \{d, f, g, i, k\}$. Now, $Sb_2(H) \leq 4$ by removing the 4 edges in $H[C]$; this removal results in vertices j, k, and l being included in every minimum dominating set as they are isolated vertices and the remaining connected component of H has a domination number of 4 and a minimum dominating set of $D' = \{d, f, g, i\}$.

It is straightforward to check that the removal of a single edge in H will not change the domination number, i.e. $b(H) \geq 2$. Therefore, by Proposition 5 we know $Sb_2(H) \geq 3$.

If $Sb_2(H) = 3$ and $b(H) \geq 2$ then there must be an edge set, E', of cardinality two and an edge ε so that $\gamma(H) = \gamma(H - E') - 1$ and $\gamma(H) = \gamma(H - E' - \{\varepsilon\}) - 2$.

It is easy to verify that the only way to increase the domination number with the removal of two edges is to remove two vertex-disjoint edges in $H[A]$. We give a brief justification below.

Consider $H[A]$, $H[C]$, and $H[B \cup F \cup G \cup \{d, i\}]$ which are three edge-disjoint subgraphs. The removal of any two edges in $H[C]$ will not increase the domination number. Also, we can see that the removal of any edge in $H[C]$ along with any other edge in H which is not in $H[C]$ will be dominated by $\{x, f', g', i, k\}$ for some $x \in A$, $f' \in F$, and $g' \in G$. The removal of any two edge in $H[F \cup G]$ will be dominated by $\{d, f', g', i, k\}$ for some vertex $f' \in F$ and $g' \in G$ because $n, m \geq 5$. The removal of any edge in $H[F]$ and any other edge in H which is not in $H[G]$ will be dominated by $\{x, f', g, i, k\}$ for some vertex $x \in A$ and $f' \in F$. Similarly, the removal of any edge in $H[G]$ and any other edge in H which is not in $H[F]$ will be dominated by $\{x, f, g', i, k\}$

for some vertex $x \in A$ and $g' \in G$. The removal of any two edges in $H[B \cup \{d, i\}]$ will be dominated by $D = \{d, f, g, i, k\}$ because e and h are adjacent to three vertices in D. And finally, the removal of an edge in $H[B \cup \{d, i\}]$ and an edge in $H[A]$ will be dominated by $\{x, f, g, i, k\}$ for some vertex $x \in A$.

Therefore, the only way to increase the domination number of H with the removal of two edges is to remove two edges in $H[A]$. For this to happen, the two edges in $H[A]$ must be vertex disjoint. Without loss of generality, we will assume we will remove $E' = \{ac, bd\}$.

Recall that the domination number can not increase by more than 1 with one edge removal by Proposition 5. Therefore, if $Sb_2(H) = 3$ there must be an edge ε in H so that $\gamma(H) = \gamma(H - \{ac, bd, \varepsilon\}) - 2$. However, a similar argument as posed for H can verify that $b(H - \{ac, bd\}) = 3$. Therefore $Sb_2(H) > 3$ and we can conclude $Sb_2(H) = 4$.

A single edge removal does not change the domination number of H and removing edges $E' = \{ac, bd\}$ increases the domination number by 1. Therefore we know $b(H) = 2$. However, $b(H - \{ac, bd\}) = 3$ which can be accomplished by the removal of $E'' = \{ad, ab, bc\}$. Thus, two successive minimum bondage moves requires the removal of 5 edges whereas $Sb_2(H) = 4$.

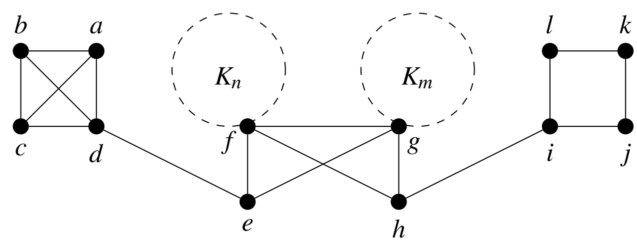

Figure 1: Graph H with $n, m \geq 4$.

Alternatively, we can show that there are circumstances under which the sum of the sizes of two successive minimum bondage moves is equal to Sb_2.

Theorem 6. *Let G be a graph. If $b(MBG(G)) \leq 2$ then $Sb_2(G) = b(G) + b(MBG(G))$.*

Proof. Since the size of two successive minimum bondage moves serves as an upper bound for $Sb_2(G)$, we will assume for the sake of contradiction that there exists a graph $G' \in \text{MBG}(G)$ so that $b(G') = b(\text{MBG}(G)) \leq 2$ and $Sb_2(G) < b(G) + b(G')$. Since $Sb_2(G) \neq b(G) + b(G')$, there exists a graph $G'' \in \text{BG}(G) - \text{MBG}(G)$ where G'' is the result of a bondage move on G of size y so that $Sb_2(G) = y + b(G'')$. Since $G'' \notin MBG(G)$ we know that $y \geq b(G) + 1$. Therefore, $Sb_2(G) \geq b(G) + 1 + b(G'')$, which leads to

$$b(G) + 1 + b(G'') \leq Sb_2(G) < b(G) + b(G'),$$

which implies

$$1 + b(G'') < b(G').$$

This yields a contradiction since $b(G'') \geq 1$, but we assumed that $b(G') \leq 2$. $\square$

Theorem 6 together with the bondage number of a path graph from [5] as provided below, assists in determining 2-synchronous bondage of path graphs.

Theorem 7. *The bondage number of a path of order $n \geq 2$ is given by*

$$b(P_n) = \begin{cases} 2, & \text{if } n \equiv 1 (\text{mod}\, 3) \\ 1, & \text{otherwise}. \end{cases}$$

A *pendant edge* is an edge that is incident to a vertex of degree 1. Next, we must show that a pendant edge in P_n is always in a minimum bondage set.

Lemma 8. *For any path graph P_n of order $n \geq 2$ with pendant edge e, there exists a minimum bondage set which contains e.*

Proof. Given any path graph, P_m, of order $m \geq 1$, $\gamma(P_m) = \lceil \frac{m}{3} \rceil$ (see [5] for example). Let e_1 be a pendant edge in P_n. We proceed by cases in using Theorem 7 to determine $b(P_n)$.

Case 1. $n \equiv 0(\text{mod } 3)$: Removing e_1 results in $P_1 \oplus P_{n-1}$. The domination number of the resulting disjoint graph is $\gamma(P_1 \oplus P_{n-1}) = \gamma(P_1) + \gamma(P_{n-1}) = 1 + \lceil \frac{n-1}{3} \rceil = 1 + \frac{n-1+1}{3} = 1 + \frac{n}{3} = 1 + \gamma(P_n)$. Since $b(P_n) = 1$ and $\gamma(P_n - \{e_1\}) = 1 + \gamma(P_n)$, the set $\{e_1\}$ is a minimum bondage set.

Case 2. $n \equiv 2(\text{mod } 3)$: Removing e_1 results in $P_1 \oplus P_{n-1}$ whose domination number is $\gamma(P_1 \oplus P_{n-1}) = \gamma(P_1) + \gamma(P_{n-1}) = 1 + \lceil \frac{n-1}{3} \rceil = 1 + \frac{n-1+2}{3} = 1 + \frac{n+1}{3} = 1 + \gamma(P_n)$. Since $b(P_n) = 1$ and $\gamma(P_n - \{e_1\}) = 1 + \gamma(P_n)$, the set $\{e_1\}$ is a minimum bondage set.

Case 3. $n \equiv 1(\text{mod } 3)$: For this case we know $n \geq 4$ and, therefore, P_n will have two pendant edges, denoted e_1 and e_2. Removing e_1 and e_2 results in $P_1 \oplus P_1 \oplus P_{n-2}$. The resulting domination number of this disjoint graph is $\gamma(P_1 \oplus P_1 \oplus P_{n-2}) = 2\gamma(P_1) + \gamma(P_{n-2}) = 2 + \lceil \frac{n-2}{3} \rceil = 1 + \frac{n+2}{3} = 1 + \gamma(P_n)$. Since $b(P_n) = 2$ and $\gamma(P_n - \{e_1, e_2\}) = 1 + \gamma(P_n)$, the set $\{e_1, e_2\}$ is a minimum bondage set. □

We now prove 2-synchronous bondage for all P_n.

Theorem 9. *For a path graph, P_n,*

$$Sb_2(P_n) = \begin{cases} 2, & n \equiv 0(\text{mod } 3) \\ 3, & \text{otherwise.} \end{cases}$$

Proof. We know from Lemma 8 that each pendant edge in P_n is contained in a minimum bondage set.

Case 1. $n \equiv 0(\text{mod } 3)$. Consider removing the two pendant edges. Let $P'_n = P_1 \oplus P_1 \oplus P_{n-2}$. So $\gamma(P'_n) = 2 + \lceil \frac{n-2}{3} \rceil = 2 + \frac{n}{3} = 2 + \gamma(P_n)$, and therefore $Sb_2(P_n) \leq 2$. By Proposition 5 we see $Sb_2(P_n) = 2$.

Case 2. $n \equiv 1(\text{mod } 3)$. We will assume that $n > 1$ because if $n = 1$ we can not increase the domination number by 2. Note that $b(P_n) = 2$ by Theorem 7. Therefore, $Sb_2(P_n) \geq 3$ by Proposition 5. Consider removing three edges, $E = \{e_1, e_2, e_3\}$, so that $P_n - E = P_1 \oplus P_1 \oplus P_1 \oplus P_{n-3}$. Note that $\gamma(P_n - E) = 3 + \gamma(P_{n-3}) = 3 + \lceil \frac{n-3}{3} \rceil = 2 + \lceil \frac{n}{3} \rceil = 2 + \gamma(P_n)$ which implies $Sb_2(P_n) \leq 3$. Therefore, we can conclude that $Sb_2(P_n) = 3$.

Case 3. $n \equiv 2(\text{mod } 3)$. We will assume that $n > 2$ because if $n = 2$ we can not increase the domination number by 2. Assume by contradiction that $Sb_2(P_n) = 2$.

By Proposition 5, there must be an edge e so that, when removed, we increase the domination number by 1. The removal of an edge from P_n will result in $P_a \oplus P_b$ where $a+b=n$. Since $n \equiv 2 (\text{mod } 3)$ then there are two possible options for the pair a,b; $a,b \equiv 1(\text{mod } 3)$ or, without loss of generality, $a \equiv 0(\text{mod } 3)$ and $b \equiv 2(\text{mod } 3)$. If $a \equiv 0(\text{mod } 3)$ and $b \equiv 2(\text{mod } 3)$ we know $\gamma(P_a \oplus P_b) = \gamma(P_a) + \gamma(P_b) = \frac{a}{3} + \lceil \frac{b}{3} \rceil = \frac{a}{3} + \frac{b+1}{3} = \frac{n+1}{3} = \lceil \frac{n}{3} \rceil$. Therefore $\gamma(P_a \oplus P_b) = \gamma(P_n)$ which tells us that the removal of the edge did not change the domination number. So $\text{MBG}(P_n) \subseteq \{P_a \oplus P_b : a,b \equiv 1(\text{mod } 3)\}$. By Theorem 7 we can conclude $b(\text{MBG}(P_n)) = 2$ since $a,b \equiv 1(\text{mod } 3)$. Since we know from Theorem 7 that $b(P_n) = 1$, Theorem 6 implies $Sb_2(P_n) = 3$. □

To conclude this section, we find bounds for general graphs based on specific characteristics including the degree of a vertex and induced subgraph structure. In [2] and [5], the authors show that the bondage number of a graph is bounded above by the minimum of one less than the sum of the degrees of two adjacent vertices. We can generalize their results to find an upper bound for $Sb_2(G)$ based on the degree of several vertices.

Theorem 10. *Let $G(V,E)$ be a graph. Then*

$$Sb_2(G) \leq min\{\deg(u) + \deg(v) + \deg(w) - \sigma(u,v,w)\},$$

where the minimum is over all sets $\{u,v,w\} \subseteq V$ where v is adjacent to both u and w, and $\sigma(u,v,w)$ is the number of edges in the induced subgraph on $\{u,v,w\}$.

Proof. Let $\{u,v,w\} \subseteq V$ be such that v is adjacent to both u and w. Let σ denote the size of the induced subgraph on u,v,w, and let $\lambda = \deg(u) + \deg(v) + \deg(w) - \sigma$. If E' is the set of edges incident to u,v, or w, then $|E'| = \lambda$.

Assume, for the sake of contradiction, that $Sb_2(G) > \lambda$. Therefore, if $G' = G - E'$, then u, v, and w are isolated in G' and $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. If D is a minimum dominating set of $G - \{u,v,w\}$, then $D \cup \{u,v,w\}$ is a minimum dominating set for G' and $D \cup \{v\}$ dominates G. Therefore $\gamma(G') = |D| + 3$ and $\gamma(G) \leq |D| + 1$ which contradicts that $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. □

Theorem 11. *Let $G(V,E)$ be a graph. Then*

$$Sb_2(G) \leq min\{\deg(u) + \deg(v) + \deg(s) + \deg(t) - 2\},$$

where the minimum is over all subsets $\{u,v,s,t\} \subseteq V$ where $uv, st \in E$ and the size of the induced subgraph on $\{u,v,s,t\}$ is two.

Proof. Let $\{u,v,s,t\} \subseteq V$ where $uv, st \in E$ and the size of the induced subgraph on $\{u,v,s,t\}$ is two. Let $\lambda = \deg(u) + \deg(v) + \deg(s) + \deg(t) - 2$. If E' is the set of edges incident to $u,v,s,$ or $,t$, then $|E'| = \lambda$.

Assume, for the sake of contradiction, that $Sb_2(G) > \lambda$. Therefore, if $G' = G - E'$, then $u,v,w,$ and t are isolated in G' and $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. If D is a minimum dominating set of $G - \{u,v,s,t\}$, then $D \cup \{u,v,s,t\}$ is a minimum dominating set for G' and $D \cup \{u,s\}$ dominates G. Therefore $\gamma(G') = |D| + 4$ and $\gamma(G) \leq |D| + 2$ which contradicts that $\gamma(G') = \gamma(G)$ or $\gamma(G') = \gamma(G) + 1$. □

4 Properties of Sb_k and Application of Sb_k to Graph Families

If a graph has multiple components, the k-synchronous bondage number can be found by taking the minimum of the sum of the l_i-synchronous bondage number for a subset of i components where $\sum l_i = k$. We note the example where $k = 2$ in the following proposition. This can be easily generalized to larger values of k.

Proposition 12. *If a graph G consists of n components $C_1, C_2, \ldots, C_n$ where $1 \leq b(C_1) \leq b(C_2) \leq \ldots \leq b(C_n)$, then*

$$Sb_2(G) = min\{Sb_2(C_1), Sb_2(C_2), \ldots, Sb_2(C_n), b(C_1) + b(C_2)\}.$$

We can also easily find the k-synchronous bondage number of a graph if there are sufficiently many pendant edges. To do so, Lemma 13 gives a sufficient condition for a vertex to be in a minimum dominating set.

Lemma 13. *Let $G = (V,E)$ be a graph with vertices $r, a, b \in V$ so that $ra, rb \in E$ and $\deg(a) = \deg(b) = 1$ (i.e. ra and rb are pendant edges). Then any minimum dominating set must contain r.*

Proof. Let $A \subseteq V$ be any minimum dominating set of G, and assume for the sake of contradiction that $r \notin A$. Then, it must be true that $a, b \in A$ since these are adjacent to no other vertices. But observe that the set $A^{'} = (A \setminus \{a,b\}) \cup \{r\}$ must also be a dominating set and $|A'| < |A|$, contradicting the minimality of A. Thus, $r \in A$. □

If we have a graph that contains a vertex that is incident to more than one pendant edge, we can find its bondage number of such a graph as shown in the following theorem.

Theorem 14. *Let $G = (V,E)$ be a graph with vertices $r, a, b \in V$ such that $ra, rb \in E$ and $\deg(a) = \deg(b) = 1$ (that is, ra and rb are pendant edges). Then, $Sb_1(G) = 1$.*

Proof. Note by Proposition 5, $Sb_1(G) \geq 1$. Next, define $G' = G - ra$, and let $A \subseteq V$ be a minimum dominating set of G. By Lemma 13 above, we know that $r \in A$. The set $A^{'} = A \cup \{a\}$ is a dominating set in G', which we claim is a minimum dominating set. To prove this, suppose for the sake of contradiction that there exists some $B^{'} \subseteq V$ so that $|B^{'}| < |A^{'}| = |A| + 1$ which dominates G'. Clearly, a must be in $B^{'}$, so if we define $B = B^{'} - \{a\}$, this is a dominating set of G with $|B| < |A|$, which was a minimal dominating set of G. This is a contradiction, so we must have that $A^{'}$ is a minimal dominating set of G', so that the bondage number of G must be one. □

Applying Theorem 14 repeatedly to a graph G which contains sufficiently many pendants, along with Proposition 5 gives the immediate result.

Corollary 15. *Let $R \subseteq V$ be such that for all $r \in R$, r is incident to at least two pendant edges in E. If $A = \{a \in V : d(a) = 1$ and a is adjacent to a vertex in $R\}$, then $Sb_{|A|-|R|}(G) = |A| - |R|$.*

Now we move on to prove the k-synchronous bondage for several well-known graph families: paths, cycles, trees, and complete graphs.

Theorem 16. *For a path graph, P_n,*

$$Sb_k(P_n) = \begin{cases} \lfloor \frac{3k-1}{2} \rfloor, & n \equiv 0 (\bmod 3) \\ \lfloor \frac{3k+1}{2} \rfloor, & n \equiv 1 (\bmod 3) \\ \lceil \frac{3k-1}{2} \rceil, & n \equiv 2 (\bmod 3). \end{cases}$$

Proof. Note that if we remove a set $\mathscr{E} \subseteq E$ from P_n then $P_n - \mathscr{E} = \bigoplus_{i=1}^{|\mathscr{E}|} P_{a_i}$, where $\sum_{i=1}^{|\mathscr{E}|+1} a_i = n$. This then gives us

$$\gamma(P_n - \mathscr{E}) \leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}|+1) \tag{1}$$

because

$$\begin{aligned} \gamma(P_n - \mathscr{E}) &= \sum_{i=1}^{|\mathscr{E}|+1} \gamma(P_{a_i}) \\ &= \sum_{i=1}^{|\mathscr{E}|+1} \left\lceil \frac{a_i}{3} \right\rceil \\ &= \sum_{a_i \equiv 0 (\bmod 3)} \frac{a_i}{3} + \sum_{a_i \equiv 1 (\bmod 3)} \frac{a_i+2}{3} + \sum_{a_i \equiv 2 (\bmod 3)} \frac{a_i+1}{3} \\ &= \sum_{i=1}^{|\mathscr{E}|+1} \frac{a_i}{3} + \sum_{a_i \equiv 1 (\bmod 3)} \frac{2}{3} + \sum_{a_i \equiv 2 (\bmod 3)} \frac{1}{3} \\ &\leq \frac{n}{3} + \sum_{a_i \equiv 1 (\bmod 3)} \frac{2}{3} + \sum_{a_i \equiv 2 (\bmod 3)} \frac{2}{3} \\ &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}|+1). \end{aligned}$$

We will proceed by cases.

Case 1: $n \equiv 0 (\bmod 3)$. First, we will show there exists a set of $\lfloor \frac{3k-1}{2} \rfloor$ edges, E', so that $\gamma(P_n - E') = \gamma(P_n) + k$. Let E' consists of the leftmost $\lfloor \frac{3k-1}{2} \rfloor$ edges of P_n. For simplicity, let $y = \lfloor \frac{3k-1}{2} \rfloor$. Thus, $P_n - E' = P_{n-y} \oplus \left(\bigoplus_{i=1}^{y} P_1 \right)$. So the domination number of $P_n - E'$ is

$$\gamma(P_n - E') = \left\lceil \frac{n-y}{3} \right\rceil + y = \left\lceil \frac{n+2y}{3} \right\rceil.$$

Consider that when k is odd, $y = \frac{3k-1}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k - \frac{1}{3} \rceil = \frac{n}{3} + k$. And when k is even, $y = \frac{3k-1}{2} - \frac{1}{2} = \frac{3k-2}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k - \frac{2}{3} \rceil = \frac{n}{3} + k$. These equalities imply that $\gamma(P_n - E') = \frac{n}{3} + k = \gamma(P_n) + k$.

Now we move to show $\lfloor \frac{3k-1}{2} \rfloor$ is the minimum number of edges which increase $\gamma(P_n)$ by k. Assume for the sake of contradiction that $Sb_k(P_n) < \lfloor \frac{3k-1}{2} \rfloor$. This implies that there is a set $\mathscr{E} \subseteq E$ such that $|\mathscr{E}| < \lfloor \frac{3k-1}{2} \rfloor$ and $\gamma(P_n - \mathscr{E}) \geq \lceil \frac{n}{3} \rceil + k = \frac{n}{3} + k$. By equation (1) we see

$$\begin{aligned}
\gamma(P_n - \mathscr{E}) &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}| + 1) \\
&\leq \frac{n}{3} + \frac{2}{3}\left(\left\lfloor \frac{3k-1}{2} \right\rfloor \right) \\
&\leq \frac{n}{3} + \frac{2}{3}\left(\frac{3k-1}{2} \right) \\
&= \frac{n}{3} + k - \frac{1}{3}.
\end{aligned}$$

But this contradicts that $\gamma(P_n - \mathscr{E}) \geq \frac{n}{3} + k$. Therefore, we have proven that $Sb_k(P_n) = \lfloor \frac{3k-1}{2} \rfloor$ for $n \equiv 0 (\text{mod } 3)$.

Case 2: $n \equiv 1(\text{mod } 3)$. Similar to the previous case, let E' consists of the leftmost $\lfloor \frac{3k+1}{2} \rfloor$ edges of P_n and let $y = \lfloor \frac{3k+1}{2} \rfloor$. Thus, $P_n - E' = P_{n-y} \oplus \left(\bigoplus_{i=1}^{y} P_1 \right)$. So the domination number of $P_n - E'$ is

$$\gamma(P_n - E') = \left\lceil \frac{n-y}{3} \right\rceil + y = \left\lceil \frac{n+2y}{3} \right\rceil.$$

When k is odd, $y = \frac{3k+1}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k + \frac{1}{3} \rceil = \frac{n+2}{3} + k$. When k is even, $y = \frac{3k}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k \rceil = \frac{n+2}{3} + k$. These equalities imply that $\gamma(P_n - E') = \frac{n+2}{3} + k = \gamma(P_n) + k$.

Now we will show $\lfloor \frac{3k+1}{2} \rfloor$ is the minimum number of edges which increase $\gamma(P_n)$ by k. Assume for the sake of contradiction that $Sb_k(P_n) < \lfloor \frac{3k+1}{2} \rfloor$. This implies that there is a set $\mathscr{E} \subseteq E$ such that $|\mathscr{E}| < \lfloor \frac{3k+1}{2} \rfloor$ and $\gamma(P_n - \mathscr{E}) \geq \lceil \frac{n}{3} \rceil + k = \frac{n+2}{3} + k$. By equation (1) we see

$$\begin{aligned}
\gamma(P_n - \mathscr{E}) &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}| + 1) \\
&\leq \frac{n}{3} + \frac{2}{3}\left(\left\lfloor \frac{3k+1}{2} \right\rfloor \right) \\
&\leq \frac{n}{3} + k + \frac{1}{3}.
\end{aligned}$$

But this contradicts that $\gamma(P_n - \mathscr{E}) \geq \frac{n+2}{3} + k$. Therefore, we have proven that $Sb_k(P_n) = \lfloor \frac{3k+1}{2} \rfloor$ for $n \equiv 1(\text{mod } 3)$.

Case 3: $n \equiv 2(\text{mod } 3)$. As before, let E' consists of the leftmost $\lceil \frac{3k-1}{2} \rceil$ edges of P_n and let $y = \lceil \frac{3k-1}{2} \rceil$. Thus, $P_n - E' = P_{n-y} \oplus \left(\bigoplus_{i=1}^{y} P_1 \right)$. So the domination number of $P_n - E'$ is

$$\gamma(P_n - E') = \left\lceil \frac{n-y}{3} \right\rceil + y = \left\lceil \frac{n+2y}{3} \right\rceil.$$

When k is odd, $y = \frac{3k-1}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k - \frac{1}{3} \rceil = \frac{n+1}{3} + k$. When k is even, $y = \frac{3k}{2}$. This implies $\left\lceil \frac{n+2y}{3} \right\rceil = \lceil \frac{n}{3} + k \rceil = \frac{n+1}{3} + k$. These equalities imply that $\gamma(P_n - E') = \frac{n+1}{3} + k = \gamma(P_n) + k$.

Now we will show $\lceil \frac{3k-1}{2} \rceil$ is the minimum number of edges which increase $\gamma(P_n)$ by k. Assume for the sake of contradiction that $Sb_k(P_n) < \lceil \frac{3k-1}{2} \rceil$. This implies that there is a set $\mathscr{E} \subseteq E$ such that $|\mathscr{E}| < \lceil \frac{3k-1}{2} \rceil$ and $\gamma(P_n - \mathscr{E}) \geq \lceil \frac{n}{3} \rceil + k = \frac{n+1}{3} + k$. By equation (1) we see

$$\begin{aligned} \gamma(P_n - \mathscr{E}) &\leq \frac{n}{3} + \frac{2}{3}(|\mathscr{E}| + 1) \\ &\leq \frac{n}{3} + \frac{2}{3}\left(\left\lceil \frac{3k-1}{2} \right\rceil\right) \\ &\leq \frac{n}{3} + \frac{2}{3}\left(\frac{3k}{2}\right) \\ &= \frac{n}{3} + k. \end{aligned}$$

But this contradicts that $\gamma(P_n - \mathscr{E}) \geq \frac{n+1}{3} + k$. Therefore, we have proven that $Sb_k(P_n) = \lceil \frac{3k-1}{2} \rceil$ for $n \equiv 2 (\text{mod } 3)$. □

It can be easily verified that for a cycle of order $n \geq 3$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ (see [5] for example). Since removing any single edge from C_n results in P_n and $\gamma(C_n) = \gamma(P_n)$ the removal of any single edge of a cycle does not change the cycle's domination number. Therefore we have the following theorem.

Theorem 17. *For a cycle graph, C_n,*

$$Sb_k(C_n) = \begin{cases} \lfloor \frac{3k-1}{2} \rfloor + 1, & n \equiv 0 (\text{mod } 3) \\ \lfloor \frac{3k+1}{2} \rfloor + 1, & n \equiv 1 (\text{mod } 3) \\ \lceil \frac{3k-1}{2} \rceil + 1, & n \equiv 2 (\text{mod } 3). \end{cases}$$

For the bondage number of trees, [5] established the following upper bound.

Theorem 18. *If T is a nontrivial tree, then $b(T) \leq 2$.*

We can extend Theorem 18 to provide a range of values for Sb_k, and these values are sharp.

Corollary 19. *Given a tree, T, with at least k edges, then $k \leq Sb_k(T) \leq 2k$ and the bounds are sharp.*

Proof. The bounds follow immediately from Proposition 5 and repeated iterations of Theorem 18. The lower bound is sharp if we consider a star graph with k edges, whose domination number is 1. To increase the domination number by k we must remove all k edges. To show that the upper bound is also sharp, we will define a special spider graph, S_k^*, as a rooted tree with $2k+2$ vertices so that the root vertex will have k children of degree 2 and one child of degree 1. Notice Figure 2 is S_2^*. Notice that $|V(S_k^*)| = 2k+2$, $|E(S_k^*)| = 2k+1$, and $\gamma(S_k^*) = k+1$. If we want to find $Sb_k(S_k^*)$ we must produce a graph that has a domination number of $2k+1$. For a graph of order n to have a domination number of $n-1$, the graph must have only 1 edge. Therefore $Sb_k(S_k^*) = 2k+1-1 = 2k$. □

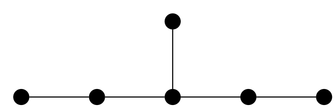

Figure 2: $Sb_2(T) = 4$.

The result for $Sb_k(K_n)$ for complete graphs stems from the following theorem from [11] which gives an upper bound for the number of edges in a graph with a specific domination number.

Theorem 20. *If G is a graph of order n and $2 \leq \gamma(G) \leq n$, then the number of edges of G is at most $\left\lfloor \frac{(n-\gamma(G))(n-\gamma(G)+2)}{2} \right\rfloor$. Equality occurs if and only if G is the disjoint union of $\gamma(G)-2$ isolated vertices and a graph obtained by removing from an $(n-\gamma(G)+2)$-clique the edges belonging to a minimum covering.*

Using Theorem 20, we present the following corollary.

Corollary 21. *For any complete graph, K_n,*

$$Sb_k(K_n) = \binom{n}{2} - \left\lfloor \frac{(n-k-1)(n-k+1)}{2} \right\rfloor.$$

Proof. From [11], we know that for any simple graph of order n and domination number d, the maximum number of edges that the graph can have is

$$\left\lfloor \frac{(n-d)(n-d+2)}{2} \right\rfloor.$$

Clearly this graph must be a subgraph of K_n. Note also that K_n has $\binom{n}{2}$ edges and $\gamma(K_n) = 1$, and so we can remove

$$\binom{n}{2} - \left\lfloor \frac{(n-k-1)(n-k+1)}{2} \right\rfloor$$

edges to leave only the subgraph mentioned in Theorem 20 with domination number $k+1$ for any positive integer k. The quantity above must be the minimum number of edges we must remove to increase the domination number of K_n by k, because if there were any smaller edge set E whose removal would increase the domination number by k, then the resulting graph $K_n - E$ would contradict Theorem 20. Thus,

$$Sb_k(K_n) = \binom{n}{2} - \left\lfloor \frac{(n-k-1)(n-k+1)}{2} \right\rfloor.$$

□

5 Conclusion

The study of dominations numbers has many applications including network design (see for example [1] and [9]). The study of bondage numbers and Sb_k are particularly important to understand the vulnerability of these networks. In this paper, we have considered applying the idea of Sb_k to sparse graphs such as paths, cycles, and trees and the extremely dense complete graph. Investigating Sb_k for additional graph families such as grid graphs or r-regular graphs would help us further understand the vulnerability of particular network structures. The interaction between Sb_k and graph operators, including disjoint unions, is another area to consider.

Furthermore, in order to create sharp bounds without regard to graph families, we must delve into the $G(n,m)$ problem; that is, given any graph on n vertices with m edges, what is the maximum number of edges that we might have to remove in order to cause a failure state? In this way, we avoid being restricted by whether a particular graph belongs to a graph family for which Sb_k has been previously determined. If we could state for certain that the removal of e edges would guarantee a failure state, we could turn our focus to efficiently determining which e edges need to be removed.

Our presentation of specific Sb_2 properties and Sb_k in general aims to extend the idea of a bondage number and thereby provide a new criteria for a failure state in a network. Since, in Section 3, we demonstrated that it is possible for a 2-synchronous bondage move to be more effective than the successive one-step counterparts, further study of Sb_k is worthwhile. To better evaluate the benefits of studying Sb_2, we would need to establish how much more efficient a 2-synchronous bondage move can be. Similarly, we would like to determine the increase in efficiency achieved by k-synchronous bondage moves when compared to successive n-synchronous bondage moves where $n < k$ and n is a factor of k.

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